STK3405 – Week 37

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Dynamic system analysis

Dynamic system analysis

Let (C, ϕ) be a binary monotone system, and introduce for $t \geq 0$:

$$\begin{aligned} X_i(t) &= \text{ the state of component } i \text{ at time } t,\ i \in C, \\ \phi(\boldsymbol{X}(t)) &= \text{ the state of the system at time } t. \end{aligned}$$

- $X_i(t)$ is a random variable (for any given t).
- $\{X_i(t)\}_{t \geq 0}$, is a stochastic process.
- $\phi(\boldsymbol{X}(t))$ is a random variable (for any given t).
- $\{\phi(\boldsymbol{X}(t))\}_{t \geq 0}$ is a stochastic process.

We assume that the stochastic processes $\{X_i(t), t \geq 0\}_{i=1}^{n}$ are independent.

Dynamic system analysis (cont.)

We also introduce:

$$p_i(t) = P(X_i(t) = 1) = \text{ The reliability of component } i \text{ at time } t,$$

$$h(\boldsymbol{p}(t)) = P(\phi(\boldsymbol{X}(t)) = 1) = \text{ The reliability of the system at time } t.$$

We assume that the components cannot be repaired and let:

$$T_i = \text{ The lifetime of component } i,$$

$$S = \text{ The lifetime of the system.}$$

NOTE:

$$P(X_i(t) = 1) = P(T_i > t), \quad i \in C,$$

$$P(\phi(\boldsymbol{X}(t)) = 1) = P(S > t).$$

Dynamic system analysis (cont.)

We denote the cumulative distribution of T_i by F_i, $i \in C$, and the cumulative distribution of ϕ by G. We then have the following relations:

$$p_i(t) = P(X_i(t) = 1) = P(T_i > t) = 1 - F_i(t) =: \bar{F}_i(t), \quad i \in C,$$

$$h(t) = P(\phi(\boldsymbol{X}(t)) = 1) = P(S > t) = 1 - G(t) =: \bar{G}(t).$$

NOTE: Determining the lifetime distribution for the system is the same as finding the reliability of the system at time t, i.e., $h(t)$, for all time $t \geq 0$, and then letting $G(t) = 1 - h(t)$.

Dynamic system analysis (cont.)

Theorem

For a monotone system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$ we have:

$$S = \begin{cases} \max_{1 \leq j \leq p} \min_{i \in P_j} T_i \\ \min_{1 \leq j \leq k} \max_{i \in K_j} T_i \end{cases}$$

PROOF: The lifetime of the system equals the lifetime of the minimal path series structure which lives the longest.

The lifetime of a minimal path series structure equals the lifetime of the shortest living component in this path set.

The second equality can be proved similarly.

Exact computation of reliability of binary monotone systems

Computational complexity

Let:

$$
\begin{aligned}
n &= \text{The size of the problem (e.g., number of components)} \\
t(n) &= \text{The worst case running time of the algorithm as a function of } n \\
f(n) &= \text{Some known non-negative increasing function of } n
\end{aligned}
$$

The order of the algorithm is said to be $O(f(n))$ if and only if there exists a positive constant M and a positive integer n_0 such that:

$$t(n) \leq Mf(n), \text{ for all } n \geq n_0.$$

If f is a polynomial in n, we say that the algorithm is a *polynomial time* algorithm, while if f is an exponential function of n, we say that the algorithm is an *exponential time* algorithm.

Computational complexity (cont.)

- NP (for nondeterministic polynomial time) is a complexity class used to describe certain types of problems.
- NP contains many important problems, the hardest of which are called *NP-complete* problems.
- Open question: Is it possible to find a polynomial time algorithm for solving NP-complete problems. Conjecture: *NO*.
- The class of *NP-hard* problems is a class of problems that are, informally, *at least as hard as the hardest problems in NP*.
- The problem of computing the reliability of a binary monotone system is known to be NP-hard in the general case.

Computational complexity (cont.)

EXAMPLE: In order to calculate the reliability of k-out-of-n system we need to do:

- $2 \cdot (2 + 3 + \cdots + n) = (n + 2)(n - 1)$ multiplications
- $1 + 2 + \cdots (n - 1) = \frac{n(n-1)}{2}$ additions

Thus, we have:

$$
\begin{aligned}
t(n) &= (n + 2)(n - 1) + \frac{n(n-1)}{2} \\
&= \frac{3}{2}n^2 + \frac{1}{2}n - 2 \\
&= 2n^2 - \frac{1}{2}(n^2 - n + 4) \leq 2n^2
\end{aligned}
$$

This shows that the reliability of a k-out-of-n system can be calculated in $O(n^2)$ time.

Threshold systems

A *threshold system* is a binary monotone system (C, ϕ), where the structure function has the following form:

$$\phi(x) = I(\sum_{i=1}^{n} a_i x_i \geq b),$$

where $a_1, \ldots, a_n$ and b are non-negative real numbers, and $I(\cdot)$ denotes the indicator function, i.e., a function defined for any event A which is 1 if A is true and zero otherwise.

NOTE: If $a_1 = \cdots = a_n = 1$ and $b = k$, the threshold system is reduced to a k-out-of-n system. Thus, threshold systems are a generalization of k-out-of-n systems.

It can be shown that calculating the reliability of a threshold system in general is NP-hard.

Threshold systems (cont.)

Let (C, ϕ) a threshold system where $a_1, \ldots, a_n$ and b are positive integers, and introduce:

$$S_j = \sum_{i=1}^{j} a_i X_i, \quad j = 1, 2, \ldots, n.$$

By the assumptions it follows that $S_1, \ldots, S_n$ are integer valued stochastic variables.

Thus, the generating function for S_j, i.e., $G_{S_j}(y) = E[y^{S_j}]$ is a polynomial, and the distribution of S_j can be derived directly from the coefficients of $G_{S_j}(y)$, $j = 1, \ldots, n$.

Threshold systems (cont.)

We also introduce:

$$d_j = \sum_{i=1}^{j} a_i, \quad j = 1, 2, \ldots, n.$$

It follows that:

$$\deg(G_{S_j}(y)) = d_j, \quad j = 1, 2, \ldots, n.$$

When $G_{S_j}(y)$ is calculated, we can find $G_{S_{j+1}}(y)$ as:

$$G_{S_{j+1}}(y) = G_{S_j}(y) \cdot G_{a_{j+1}X_{j+1}}(y)$$

In the worst case this would require $2(d_j + 1)$ multiplications and d_j additions.

Threshold systems (cont.)

EXAMPLE: Assume that $a_j = 2^{j-1}$, $j = 1, \ldots, n$. We then have:

$$\deg(G_{S_j}(y)) = d_j = \sum_{i=1}^{j} 2^{i-1} = 2^j - 1, \quad j = 1, 2, \ldots, n.$$

In fact, in this case $G_{S_j}(y)$ consists of 2^j non-zero terms (including the constant term)!

Calculating $G_{S_{j+1}}(y)$ from $G_{S_j}(y)$ would require 2^{j+1} multiplications and $2^j - 1$ additions.

Thus, using generating functions for calculating the reliability of this threshold system takes $O(2^n)$ time.

Threshold systems (cont.)

EXAMPLE: Assume that $a_j \leq A$, $j = 1, \ldots, n$, where A is a fixed positive integer. We then have:

$$\deg(G_{S_j}(y)) = d_j \leq \sum_{i=1}^{j} A = Aj, \quad j = 1, 2, \ldots, n.$$

Calculating $G_{S_{j+1}}(y)$ from $G_{S_j}(y)$ would require at most $2(Aj + 1)$ multiplications and Aj additions.

Since A is a fixed constant, it follows that calculating the reliability of such a threshold system takes $O(n^2)$ time.

Section 4.1.

State space enumeration

State space enumeration

We know that the reliability of a binary monotone system (C, ϕ) is simply the expected value of $\phi(\boldsymbol{X})$.

Thus, by standard probability theory this can be calculated as:

$$h = E[\phi(\boldsymbol{X})] = \sum_{\boldsymbol{X} \in \{0,1\}^n} \phi(\boldsymbol{x}) P(\boldsymbol{X} = \boldsymbol{x})$$

NOTE:

- Calculating reliability using state space enumeration implies summing 2^n terms.
- If the components are dependent, the entire joint distribution of $\boldsymbol{X}$ is needed.

State space enumeration (cont.)

If the component reliabilities are $p_1, \ldots, p_n$, we may write:

$$P(X_i = x_i) = p_i^{x_i}(1 - p_i)^{1-x_i}, \quad x_i = 0, 1, \quad i = 1, \ldots, n.$$

Thus, if the components are independent, we have:

$$P(\boldsymbol{X} = \boldsymbol{x}) = \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}$$

Inserting this into the state space enumeration formula we get:

$$h(\boldsymbol{p}) = \sum_{\boldsymbol{X} \in \{0,1\}^n} \phi(\boldsymbol{x}) \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}.$$

State space enumeration (cont.)

NOTE: Due to the large number of terms in the expansion, the algorithm is at least of order $O(2^n)$. The order is typically greater since the work of computing $\phi(\cdot)$ for each $\boldsymbol{x}$ must be included as well.

It is possible to improve the algorithm by utilizing that ϕ is non-decreasing in each argument:

If e.g., $\phi(\boldsymbol{x}_0) = 0$, it follows that $\phi(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} < \boldsymbol{x}_0$ as well. Hence, any state vector such that $\boldsymbol{x} < \boldsymbol{x}_0$ can be eliminated from the sum.

State space enumeration (cont.)

Let $C = \{1, 2, 3\}$, and let:

$$\phi(\boldsymbol{x}) = (x_1 \amalg x_2) \cdot x_3.$$

The component state variables are independent, and the component reliabilities are $P(X_i = 1) = p_i$, $i = 1, 2, 3$.

We then have:

$$\begin{aligned} h(\boldsymbol{p}) &= \phi(0,0,0) \cdot (1 - p_1)(1 - p_2)(1 - p_3) \\ &\quad + \phi(1,0,0) \cdot p_1(1 - p_2)(1 - p_3) + \cdots + \phi(1,1,1) \cdot p_1 p_2 p_3 \\ &= p_1(1 - p_2)p_3 + (1 - p_1)p_2 p_3 + p_1 p_2 p_3. \end{aligned}$$

State space enumeration (cont.)

NOTE: There are exactly 3 vectors such that $\phi(\boldsymbol{x}) = 1$:

$$\boldsymbol{x}_1 = (1,0,1), \quad \boldsymbol{x}_2 = (0,1,1), \quad \boldsymbol{x}_3 = (1,1,1).$$

Hence, the reliability function contains just 3 terms, not $2^3 = 8$ terms.

NOTE: All terms have p_3 as a common factor, and:

$$p_1(1-p_2) + (1-p_1)p_2 + p_1 p_2 = p_1 \amalg p_2$$

Hence, $h(\boldsymbol{p})$ can be simplified to:

$$\begin{aligned} h(\boldsymbol{p}) &= [p_1(1-p_2) + (1-p_1)p_2 + p_1 p_2] \cdot p_3 \\ &= (p_1 \amalg p_2) p_3 \end{aligned}$$

The last expression can be obtained directly from the structure function since the system is an s-p-system.

The multiplication method

The multiplication method

Consider a binary monotone system with *minimal path sets*:

$$P_1, \dots, P_p,$$

and *minimal cut sets*:

$$K_1, \dots, K_k.$$

We then have:

$$\begin{aligned}\phi(\boldsymbol{X}) &= \coprod_{j=1}^{p} \prod_{i \in P_j} X_i = 1 - [\prod_{j=1}^{p}(1 - \prod_{i \in P_j} X_i)] \\ &= \prod_{j=1}^{k} \coprod_{i \in K_j} X_i = \prod_{j=1}^{k}[1 - \prod_{i \in K_j}(1 - X_i)].\end{aligned}$$

The multiplication method (cont.)

By expanding either the formula based on the minimal path sets, or the formula based on the minimal cut sets, and using that $X_i^r = X_i$, $i = 1, \ldots, n$, $r = 1, 2, \ldots$, we eventually get an expression of the form:

$$\phi(\boldsymbol{X}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} X_i$$

where for all $A \subseteq C$, $\delta(A)$ denotes the coefficient of the term associated with $\prod_{i \in A} X_i$. The δ-function is called the *signed domination function* of the structure.

The multiplication method (cont.)

By taking the expectation on both sides, and assuming that the component state variables are independent, we obtain:

$$h(\boldsymbol{p}) = E[\phi(X)] = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} E[X_i] = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} p_i$$

The sum contains 2^n terms.

Hence, the multiplication method also has at least order $O(2^n)$.

Typically, many of the terms vanish since we may have $\delta(A) = 0$ for many of the sets A. Improved versions of this method utilizes this.

The multiplication method (cont.)

Introduce the following version of the structure function of a binary monotone system (C, ϕ):

$$\phi(B) = \phi(1^B, 0^{\bar{B}}), \text{ for all } B \subseteq C.$$

Thus, $\phi(B)$ represents the state of the system given that all the components in the set B are functioning, while all the components in the set $\bar{B} = C \setminus B$ are failed.

We now have:

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B), \text{ for all } A \subseteq C.$$

NOTE: This formula allows us to compute the signed domination function without using the minimal path and cut sets. For certain classes of systems this can be used to establsih formulas allowing much faster calculations.

Properties of the signed domination

Theorem

Let (C, ϕ) be a binary monotone system, and let $A \subseteq C$.

If $\phi(A) = 0$, then $\delta(A) = 0$ as well.

PROOF: Since ϕ is nondecreasing in each argument, it follows from the assumption that $\phi(B) = 0$ for all $B \subseteq A$. Hence,

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B) = 0.$$

Properties of the signed domination (cont.)

Theorem

Let (C, ϕ) be a binary monotone system, let $A \subseteq C$, and let $i \in A$. If $\phi(B \cup i) = \phi(B)$ for all $B \subseteq A \setminus i$, then $\delta(A) = 0$.

PROOF:

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B)$$

$$= \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B|} \phi(B) + \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B \cup i|} \phi(B \cup i)$$

$$= \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B|} \phi(B) - \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B|} \phi(B) = 0$$

Properties of the signed domination (cont.)

Theorem

Let (C, ϕ) be a binary monotone system, let $P \subseteq C$ be a minimal path set.

Then $\delta(P) = 1$.

PROOF: Since P is a minimal path set, it follows that $\phi(P) = 1$ and that $\phi(B) = 0$ for all $B \subset P$. Hence,

$$\delta(P) = \sum_{B \subseteq P} (-1)^{|P|-|B|} \phi(B)$$

$$= (-1)^{|P|-|P|} \phi(P) = 1.$$

The multiplication method (cont.)

Consider a binary monotone system (C, ϕ) where $C = \{1, 2, 3\}$, with minimal path sets $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$. Using the multiplication method we obtain:

$$\begin{aligned}
\phi(\boldsymbol{X}) &= (X_1 X_2) \amalg (X_1 X_3) = 1 - (1 - X_1 X_2)(1 - X_1 X_3) \\
&= 1 - (1 - X_1 X_2 - X_1 X_3 + X_1^2 X_2 X_3) \\
&= X_1 X_2 + X_1 X_3 - X_1 X_2 X_3,
\end{aligned}$$

where we have used that $X_1^2 = X_1$.

Thus, $\delta(\{1, 2\}) = \delta(\{1, 3\}) = 1$, $\delta(\{1, 2, 3\}) = -1$, while $\delta(A) = 0$ for all other subsets of C.

The multiplication method (cont.)

Note that these coefficients can be obtained using the signed domination formula and the theorems as well since:

$$\delta(\{1,2\}) = 1, \quad \text{since } \{1,2\} \text{ is a minimal path set}$$

$$\delta(\{1,3\}) = 1, \quad \text{since } \{1,3\} \text{ is a minimal path set}$$

$$\begin{aligned}\delta(\{1,2,3\}) = (-1)^{|\{1,2,3\}|-|\{1,2\}|}\phi(\{1,2\}) + (-1)^{|\{1,2,3\}|-|\{1,3\}|}\phi(\{1,2\}) \\ + (-1)^{|\{1,2,3\}|-|\{1,2,3\}|}\phi(\{1,2,3\}) = -1 - 1 + 1 = -1.\end{aligned}$$

Having derived the formula for the structure function ϕ, we immediately obtain the reliability function:

$$h(\boldsymbol{p}) = p_1 p_2 + p_1 p_3 - p_1 p_2 p_3.$$

Section 4.3.

The inclusion-exclusion method

The inclusion-exclusion method

Consider a binary monotone system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$. We then introduce the events

$$E_j = \{\text{All of the components in } P_j \text{ are functioning}\}, \quad j = 1, \ldots, p.$$

Since the system is functioning if and only if at least one of the minimal path sets is functioning, we have:

$$\phi = 1 \quad \text{if and only if} \quad \bigcup_{j=1}^{p} E_j \text{ holds true.}$$

The inclusion-exclusion method (cont.)

We then use the *inclusion-exclusion* formula and get:

$$
\begin{aligned}
h = P(\bigcup_{j=1}^{p} E_j) \\
&= P(E_1) + P(E_2) + \cdots + P(E_p) \\
&- P(E_1 \cap E_2) - P(E_1 \cap E_3) - \cdots - P(E_{p-1} \cap E_p) \\
&+ P(E_1 \cap E_2 \cap E_3) + \cdots + P(E_{p-2} \cap E_{p-1} \cap E_p) \\
&\cdots \\
&+ (-1)^{p-1} P(E_1 \cap \cdots \cap E_p).
\end{aligned}
$$

NOTE: The number of terms is $2^p - 1$. However, typically many of the terms can be merged as they correspond to the same component set.

The inclusion-exclusion method (cont.)

NOTE: An event of form $E_{i_1} \cap \cdots \cap E_{i_r}$ occurs if and only if all the components in the set $P_{i_1} \cup \cdots \cup P_{i_r}$ are functioning. When the component state variables are independent, we get:

$$P(E_{i_1} \cap \cdots \cap E_{i_r}) = \prod_{i \in P_{i_1} \cup \cdots \cup P_{i_r}} p_i.$$

The inclusion-exclusion method (cont.)

Consider a binary monotone system (C, ϕ) where $C = \{1, 2, 3, 4\}$ with minimal path sets $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$, $P_3 = \{2, 3, 4\}$.

Assuming that the component state variables are independent, we then get:

$$P(E_1) = p_1 p_2, \quad P(E_2) = p_1 p_3, \quad P(E_3) = p_2 p_3 p_4$$
$$P(E_1 \cap E_2) = p_1 p_2 p_3, \quad P(E_1 \cap E_3) = P(E_2 \cap E_3) = p_1 p_2 p_3 p_4$$
$$P(E_1 \cap E_2 \cap E_3) = p_1 p_2 p_3 p_4.$$

Hence, the reliability of the system is:

$$\begin{aligned} h(\boldsymbol{p}) &= p_1 p_2 + p_1 p_3 + p_2 p_3 p_4 - p_1 p_2 p_3 - 2 p_1 p_2 p_3 p_4 + p_1 p_2 p_3 p_4 \\ &= p_1 p_2 + p_1 p_3 + p_2 p_3 p_4 - p_1 p_2 p_3 - p_1 p_2 p_3 p_4. \end{aligned}$$

The inclusion-exclusion method (cont.)

NOTE: The final expression for the reliability function is exactly the same as we get using the multiplication methods:

$$h(\boldsymbol{p}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} p_i,$$

where δ denotes the signed domination function. However, the steps we take in order to get this expression is different.

By using the inclusion-exclusion formula we see that all terms in the expansion correspond to sets $A \subseteq C$ which are unions of minimal path sets.

If a set $A \subseteq C$ is *not* a union of minimal path sets, it follows that $\delta(A) = 0$.

The inclusion-exclusion method (cont.)

If $P_{i_1}, \ldots, P_{i_r}$ is a collection of minimal path sets such that:

$$\bigcup_{j=1}^{r} P_{i_j} = A,$$

the collection is said to be a *formation* of the set A.

The formation is *odd* if r is odd, and *even* if r is even.

An odd formation contributes with a coefficient $+1$, while an even formation contributes with a coefficient -1.

The inclusion-exclusion method (cont.)

Simplifying the expansion, all terms corresponding to formations of the same set are merged. Hence, it follows that we have:

Theorem

Let (C, ϕ) be a binary monotone system with minimal path sets $P_1, \ldots, P_p$, and let δ denote the signed domination function of the system. Then for all $A \subseteq C$ we have:

$$\begin{aligned}\delta(A) = &\ \textit{The number of odd formations of } A \\ &- \textit{The number of even formations of } A.\end{aligned}$$

In particular $\delta(A) = 0$ if A is not a union of minimal path sets.

As a corollary we obtain the following result:

Corollary

If (C, ϕ) is a binary monotone system which is not coherent, then $\delta(C) = 0$.

An upper bound on the system reliability

Skipping all higher order terms and keeping the probabilities of the individual events $E_1, \ldots, E_p$ only, we get an *upper bound* on the system reliability:

$$h \leq \sum_{j=1}^{p} P(E_j).$$

Given that p is moderate, and that we are given the minimal path sets, this upper bound is easy to calculate.

An lower bound on the system reliability

We then consider the minimal cut sets of the system $K_1, \ldots, K_k$, and introduce:

$$F_j = \{\text{All the components in } K_j \text{ are failed}\}, \quad j = 1, \ldots, k.$$

Since the system is failed if and only if all the components in at least one cut set are failed, we have:

$$1 - h = P(\bigcup_{j=1}^{k} F_j).$$

An upper bound on $1 - h$ is then given by:

$$1 - h \leq \sum_{j=1}^{k} P(F_j).$$

Combining the upper and lower bounds, we get:

$$1 - \sum_{j=1}^{k} P(F_j) \leq h \leq \sum_{j=1}^{p} P(E_j).$$

Bounds on the system reliability (cont.)

If the components are independent, we get:

$$1 - \sum_{j=1}^{k} \prod_{i \in K_j} (1 - p_i) \leq h(\mathbf{p}) \leq \sum_{j=1}^{p} \prod_{i \in P_j} p_i.$$

If the component reliabilites are close to 1, the lower bound turns out to be very good, while the upper bound will be crude and possibly greater than 1.

If the component reliabilites are close to 0, the lower bound will be crude and possibly less than 0, while the upper bound turns out to be very good.

Bounds on the system reliability (cont.)

In order to avoid bounds outside of the interval $[0, 1]$, one would typically replace the bounds by:

$$\max(1 - \sum_{j=1}^{k} \prod_{i \in K_j} (1 - p_i),\ 0) \leq h(\mathbf{p}) \leq \min(\sum_{j=1}^{p} \prod_{i \in P_j} p_i,\ 1).$$