STK3405 – Week 44

A. B. Huseby & K. R. Dahl

Department of Mathematics
University of Oslo, Norway

Pure jump processes

Let $\{S(t)\}$ be a stochastic process where $S(t)$ denotes the state of the process at time $t \geq 0$. $\{S(t)\}$ is said to be a *pure jump process* if $S(t)$ can be written as:

$$S(t) = S(0) + \sum_{j=1}^{\infty} I(T_j \leq t) J_j, \qquad t \geq 0,$$

where $0 = T_0 < T_1 < T_2 < \cdots$ is a sequence of random points of time, and $J_1, J_2, \ldots$ is a sequence of random *jumps*.

This representation implies that the state function $S(t)$ is *piecewise constant* and *right-continuous* in t, with jumps at $T_1 < T_2 < \cdots$.

In particular, for $k = 0, 1, \ldots$, we have:

$$S(t) = S(0) + \sum_{j=1}^{k} J_j = S(T_k), \qquad \text{for all } t \in [T_k, T_{k+1}).$$

In order to keep track of how the process evolves only the *event points* need to be considered.

Regular pure jump processes

Let $\{S(t)\}$ be pure jump process, and let:

$$N(t) = \sum_{j=1}^{\infty} I(T_j \leq t)$$
$$= \text{The number of jumps in [0, t].}$$

We say that $\{S(t)\}$ is *regular* if $P(N(t) < \infty) = 1$ for all $t > 0$.

NOTE

$$P(N(t) < \infty) = P(\lim_{k \to \infty} T_k = \infty)$$
$$= P(\lim_{k \to \infty} \sum_{j=1}^{k} \Delta_j = \infty),$$

where $\Delta_j = T_j - T_{j-1}$, $j = 1, 2, \ldots$.

Regular pure jump processes (cont.)

Proposition (8.1.1)

Let $\{S(t)\}$ be a pure jump process with jumps at:

$$T_1 < T_2 < \cdots$$

Moreover, we let $T_0 = 0$ and introduce the non-negative random variables $\Delta_j = T_j - T_{j-1}$, $j = 1, 2, \ldots$.

If the sequence $\{\Delta_j\}$ contains an infinite subsequence $\{\Delta_{k_j}\}$ of independent, identically distributed random variables such that $E[\Delta_{k_j}] = d > 0$, then S is regular.

Regular pure jump processes (cont.)

PROOF: By the strong law of large numbers it follows that:

$$P(\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \Delta_{k_j} = d) = 1.$$

This implies that the series $\sum_{j=1}^{\infty} \Delta_{k_j}$ is divergent with probability one.

Hence, since obviously $\sum_{j=1}^{\infty} \Delta_{k_j} \leq \sum_{j=1}^{\infty} \Delta_j$, the result follows.

Regular pure jump processes (cont.)

Proposition (8.1.2)

Let $\{S(t)\}$ *be a regular pure jump process with jumps at* $T_1 < T_2 < \cdots$*. Then* $\lim_{t \to s^-} S(t)$ *exists for every* $s > 0$ *with probability one.*

PROOF: Let $0 \leq t < s < \infty$, and consider the set:

$$\mathcal{T} = \{T_j : t \leq T_j < s\} \cup \{t\}.$$

Since S is regular, the set $\mathcal{T}$ is finite with probability one. Moreover, $\mathcal{T}$ is non-empty since $t \in \mathcal{T}$. Thus, this set contains a maximal element, which we denote by t'. Moreover, since every element in $\mathcal{T}$ is less than s, then so is t'.

From this it follows that the interval (t', s) is nonempty.

At the same time (t', s) does not contain any jumps, so $S(t)$ is constant throughout this interval. Hence, $\lim_{t \to s^-} S(t)$ exists. Since s was arbitrary chosen, this holds for any $s > 0$.

Regular pure jump processes (cont.)

Proposition (8.1.3)

Let $\{S(t)\}$ be a regular pure jump process with jumps at $T_1 < T_2 < \cdots$, and let $0 \leq u < v < \infty$.

Assume that $\{T_j : u < T_j < v\} = \{T^{(1)}, \ldots, T^{(k)}\}$, where $T^{(1)} < \cdots < T^{(k)}$.

Moreover, we define $T^{(0)} = u$ and $T^{(k+1)} = v$.

Then we have:

$$\int_u^v S(t)dt = \sum_{j=0}^{k} S(T^{(j)})(T^{(j+1)} - T^{(j)}).$$

Regular pure jump processes (cont.)

NOTE: Since S is regular, the number of elements in the set $\{T_j : u < T_j < v\}$ is finite with probability one.

Thus, this set can almost surely be written in the form $\{T^{(1)}, \ldots, T^{(k)}\}$, for some suitable $k < \infty$.

Since S is right-continuous and piecewise constant, it follows that $S(t) = S(T^{(j)})$ for all $t \in [T^{(j)}, T^{(j+1)})$, $j = 0, 1, \ldots, k$.

Thus, we have:

$$\int_{T^{(j)}}^{T^{(j+1)}} S(t)dt = S(T^{(j)})(T^{(j+1)} - T^{(j)}), \quad j = 0, 1, \ldots, k.$$

The result then follows by adding up the contributions to the integral from each of the $k + 1$ intervals $[T^{(0)}, T^{(1)}), \ldots, [T^{(k)}, T^{(k+1)})$

Regular pure jump processes (cont.)

Proposition (8.1.4)

Let $\{S_1(t)\}, \ldots, \{S_n(t)\}$ be n regular pure jump processes, and let $H(t) = H(\boldsymbol{S}(t))$, where $\boldsymbol{S}(t) = (S_1(t) \ldots, S_n(t))$, $t \geq 0$.

Then $\{H(t)\}$ is a regular pure jump process as well.

That is, $H(t) = H(\boldsymbol{S}(t))$ is:

- *Piecewise constant*
- *Right-continuous in t,*
- *The number of jumps in $[0, t]$ is finite with probability one for all $t > 0$*

Regular pure jump processes (cont.)

PROOF: Let $\mathcal{T}_i$ be the set of jump points of S_i, $i = 1, \ldots, n$, and let $\mathcal{T}$ be the set of jump points of the process $\{H(t)\}$.

Since the state value of H cannot change unless there is a change in the state value of at least one of the elementary processes, it follows that $\mathcal{T} \subseteq (\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n)$.

Thus, $H(t)$ is piecewise constant and right-continuous in t. Moreover, for any finite interval $[0, t]$ we also have:

$$\mathcal{T} \cap [0, t] \subseteq [(\mathcal{T}_1 \cap [0, t]) \cup \cdots \cup (\mathcal{T}_n \cap [0, t])].$$

By regularity $(\mathcal{T}_i \cap [0, t])$ is finite almost surely for $i = 1, \ldots, n$. Hence, $\mathcal{T} \cap [0, t]$ is finite almost surely as well, implying that $\{H(t)\}$ is regular.

Application to reliability

Consider (C, ϕ), a binary monotone system of *n repairable* components.

Component state processes: $\{X_1(t)\}, \ldots, \{X_n(t)\}$, where:

$$X_i(t) = \text{the state of component } i \text{ at time } t \geq 0, \quad i \in C.$$

Component state vector: $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$

System state process: $\{\phi(t)\}$, where:

$$\phi(t) = \phi(\boldsymbol{X}(t)) = \text{the state of the system at time } t \geq 0$$

Repairable components

We introduce the following random variables:

- U_{ij} = The jth lifetime of the ith component
- D_{ij} = The jth repair time of the ith component

For $i = 1, \ldots, n$, $j = 1, 2, \ldots$:

- U_{ij} has an abs. continuous distribution with mean value $0 < \mu_i < \infty$
- D_{ij} has an abs. continuous distribution with mean value $0 < \nu_i < \infty$

All lifetimes and repair times are assumed to be *independent*. Thus, the component processes $\{X_1(t)\}, \ldots, \{X_n(t)\}$ are independent of each other.

Repairable components

Now we let for $i = 1, \ldots, n$:

$$\begin{aligned} T_{i,1} &= U_{i1}, \\ T_{i,2} &= U_{i1} + D_{i1} \\ T_{i,3} &= U_{i1} + D_{i1} + U_{i2}, \\ T_{i,4} &= U_{i1} + D_{i1} + U_{i2} + D_{i2} \\ &\cdots \end{aligned}$$

Moreover, let $J_j = (-1)^j$. We may then write:

$$X_i(t) = X(0) + \sum_{j=1}^{\infty} I(T_{ij} \leq t) J_j, \quad i = 1, \ldots, n.$$

By Proposition 8.1.1 $\{X_1(t)\}, \ldots, \{X_n(t)\}$ are regular pure jump processes.

By Proposition 8.1.4 $\{\phi(t)\}$ is a regular pure jump process as well.

Availability

Let $A_i(t)$ be the availability of the ith component at time t. That is, for $i = 1, \ldots, n$ we have:

$$A_i(t) = \Pr(X_i(t) = 1) = E[X_i(t)].$$

By renewal theory the corresponding stationary availabilities are given by:

$$A_i = \lim_{t \to \infty} A_i(t) = \frac{\mu_i}{\mu_i + \nu_i}, \qquad i = 1, \ldots, n.$$

Introduce $\boldsymbol{A}(t) = (A_1(t), \ldots, A_n(t))$ and $\boldsymbol{A} = (A_1, \ldots, A_n)$. The system availability at time t is given by:

$$A_\phi(t) = \Pr(\phi(\boldsymbol{X}(t)) = 1) = E[\phi(\boldsymbol{X}(t))] = h(\boldsymbol{A}(t)),$$

where h is the system's reliability function. The corresponding stationary availability is given by:

$$A_\phi = \lim_{t \to \infty} A_\phi(t) = h(\boldsymbol{A})$$

Criticality

The component i is said to be *critical* at time t if

$$\psi_i(\boldsymbol{X}(t)) = \phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t)) = 1.$$

$\psi_i(\boldsymbol{X}(t))$ is the *criticality state* of component i at time t.

The Birnbaum measure of importance of component i at time t, $I_B^{(i)}(t)$, is the probability that i is critical at time t:

$$\begin{aligned} I_B^{(i)}(t) &= \Pr(\psi_i(\boldsymbol{X}(t)) = 1) = E[\psi_i(\boldsymbol{X}(t))] \\ &= h(1_i, \boldsymbol{A}(t)) - h(0_i, \boldsymbol{A}(t)). \end{aligned}$$

The corresponding stationary measure is given by:

$$I_B^{(i)} = \lim_{t \to \infty} I_B^{(i)}(t) = h(1_i, \boldsymbol{A}) - h(0_i, \boldsymbol{A}).$$