# STK3405 – Week 45

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# Application to reliability

Consider $(C, \phi)$, a binary monotone system of *n repairable* components.

**Component state processes**: $\{X_1(t)\}, \ldots, \{X_n(t)\}$, where:

$$X_i(t) = \text{the state of component } i \text{ at time } t \geq 0, \quad i \in C.$$

**Component state vector**: $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$

**System state process**: $\{\phi(t)\}$, where:

$$\phi(t) = \phi(\boldsymbol{X}(t)) = \text{the state of the system at time } t \geq 0$$

# Repairable components

We introduce the following random variables:

- $U_{ij}$ = The $j$th lifetime of the $i$th component
- $D_{ij}$ = The $j$th repair time of the $i$th component

For $i = 1, \ldots, n$, $j = 1, 2, \ldots$:

- $U_{ij}$ has an abs. continuous distribution with mean value $0 < \mu_i < \infty$
- $D_{ij}$ has an abs. continuous distribution with mean value $0 < \nu_i < \infty$

All lifetimes and repair times are assumed to be *independent*. Thus, the component processes $\{X_1(t)\}, \ldots, \{X_n(t)\}$ are independent of each other.

# Availability

Let $A_i(t)$ be the availability of the $i$th component at time $t$. That is, for $i = 1, \ldots, n$ we have:

$$A_i(t) = \Pr(X_i(t) = 1) = E[X_i(t)].$$

By renewal theory the corresponding stationary availabilities are given by:

$$A_i = \lim_{t \to \infty} A_i(t) = \frac{\mu_i}{\mu_i + \nu_i}, \qquad i = 1, \ldots, n.$$

Introduce $\boldsymbol{A}(t) = (A_1(t), \ldots, A_n(t))$ and $\boldsymbol{A} = (A_1, \ldots, A_n)$. The system availability at time $t$ is given by:

$$A_\phi(t) = \Pr(\phi(\boldsymbol{X}(t)) = 1) = E[\phi(\boldsymbol{X}(t))] = h(\boldsymbol{A}(t)),$$

where $h$ is the system's reliability function. The corresponding stationary availability is given by:

$$A_\phi = \lim_{t \to \infty} A_\phi(t) = h(\boldsymbol{A})$$

# Criticality

The component $i$ is said to be *critical* at time $t$ if

$$\psi_i(\boldsymbol{X}(t)) = \phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t)) = 1.$$

$\psi_i(\boldsymbol{X}(t))$ is the *criticality state* of component $i$ at time $t$.

The Birnbaum measure of importance of component $i$ at time $t$, $I_B^{(i)}(t)$, is the probability that $i$ is critical at time $t$:

$$\begin{aligned} I_B^{(i)}(t) &= \Pr(\psi_i(\boldsymbol{X}(t)) = 1) = E[\psi_i(\boldsymbol{X}(t))] \\ &= h(1_i, \boldsymbol{A}(t)) - h(0_i, \boldsymbol{A}(t)). \end{aligned}$$

The corresponding stationary measure is given by:

$$I_B^{(i)} = \lim_{t\to\infty} I_B^{(i)}(t) = h(1_i, \boldsymbol{A}) - h(0_i, \boldsymbol{A}).$$

# Event model

Let $(C, \phi)$ be a binary monotone system with component state processes: $\{X_1(t)\}, \ldots, \{X_n(t)\}$.

- $E_{i1}, E_{i2}, \ldots$ are the events affecting the process $\{X_i(t)\}$
- $T_{i1}, T_{i2}, \ldots$ are the corresponding points of time for these events

Since we assumed that all lifetimes and repair times have *absolutely continuous distributions*, all the events happen at *distinct* points of time almost surely, i.e., all the $T_{ij}$s are distinct numbers.

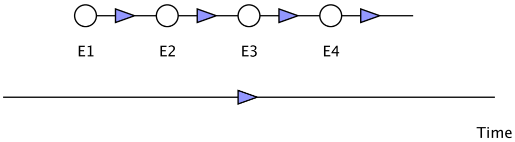
We assume that the events are sorted with respect to their respective points of time, so that $T_{i1} < T_{i2} < \cdots$.

# Event model (cont.)

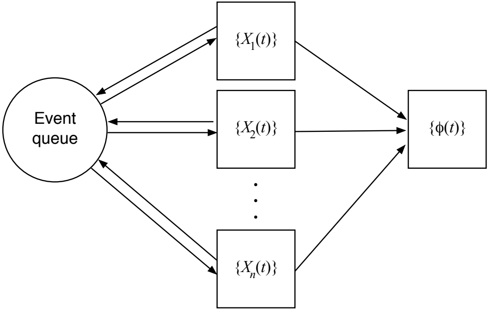
At the system level the event set is the *union* of all the component event sets.

- Let $E^{(1)}, E^{(2)}, \ldots$ denote the system events sorted with respect to their respective points of time
- Let $T^{(1)} < T^{(2)} < \cdots$ be the corresponding points of time

Each system event corresponds to a unique component event, organized in a dynamic queue sorted with respect to the points of time of the events:



# Program flow




- The components post initial events to the event queue
- The event queue processes events in chronological order, and notifies the components when the events occur. As soon as an event is processed, it is removed from the queue.
- The component updates its state, posts a new event to the queue, and notifies the system about the state change

# Sampling events

Although the system state and component states stay constant between events, it is of interest to sample the state values at *predefined* points of time. Thus, we introduce yet another type of event, called a *sampling events* spread out evenly on the timeline.

- Let $e_1, e_2, \ldots$ denote the sampling events
- Let $t_1 < t_2 < \cdots$ are the corresponding points of time

Typically $t_j = j \cdot \Delta$ for some suitable $\Delta > 0$, $j = 1, 2, \ldots$.

The sampling events are placed into the queue in the same way as for the ordinary events.

# Pointwise estimates of availability and importance

**Goal**: Estimate $A_\phi(t)$ and $I_B^{(1)}(t), \ldots, I_B^{(n)}(t)$ for $0 \leq t \leq t_N$

**Solution**: Pointwise estimates of $A_\phi(t)$ and $I_B^{(1)}(t), \ldots, I_B^{(n)}(t)$ for $t \in \{t_1, \ldots, t_N\}$, and use interpolation between these points.

In each simulation we sample the values of $\phi$ and $\psi_1, \ldots, \psi_n$ at each sampling point $t_1, \ldots, t_N$. We denote the $s$th simulated result of the component state vector process by $\{\boldsymbol{X}_s(t)\}$, $s = 1, \ldots, M$, and obtain the following estimates for $j = 1, \ldots, N$:

$$\hat{A}_\phi(t_j) = \frac{1}{M} \sum_{s=1}^{M} \phi(\boldsymbol{X}_s(t_j)),$$

$$\hat{I}_B^{(i)}(t_j) = \frac{1}{M} \sum_{s=1}^{M} \psi_i(\boldsymbol{X}_s(t_j)).$$

# Interval estimates of availability and importance

**Alternative idea**: Use average simulated availability and criticalities from each interval $(t_{j-1}, t_j]$, $j = 1, \ldots, N$ as estimates for the availability and criticalities at the midpoints of these intervals.

We then obtain the following estimates for $j = 1, \ldots, N$:

$$\tilde{A}_\phi(\bar{t}_j) = \frac{1}{M} \sum_{s=1}^{M} \frac{1}{\Delta} \sum_{k \in \mathcal{E}_{sj}} \phi(\boldsymbol{X}_s(T_s^{(k)}))(T_s^{(k+1)} - T_s^{(k)}),$$

$$\tilde{I}_B^{(i)}(\bar{t}_j) = \frac{1}{M} \sum_{s=1}^{M} \frac{1}{\Delta} \sum_{k \in \mathcal{E}_{sj}} \psi_i(\boldsymbol{X}_s(T_s^{(k)}))(T_s^{(k+1)} - T_s^{(k)}),$$

where $\mathcal{E}_{sj}$ denotes the index set of the events in $(t_{j-1}, t_j]$ in the $s$th simulation, and $\bar{t}_j = (t_{j-1} + t_j)/2$.

# Interval estimates of availability and importance

- The integral formula given in one of the propositions implies that $\tilde{A}_\phi(\bar{t}_j)$ and $\tilde{I}_B^{(i)}(\bar{t}_j)$ are unbiased and strongly consistent estimates of the corresponding average availability and criticality in the intervals $[t_{j-1}, t_j)$ respectively.
- By choosing $\Delta$ so that the availabilities and criticalities are relatively stable within each interval, the interval estimates are approximately unbiased estimates for $A_\phi(\bar{t}_j)$ and $I_B^{(i)}(\bar{t}_j)$ as well.
- The resulting interval estimates stabilize much faster than the pointwise estimates.
- Interpolation is used to estimate $A_\phi(t)$ and $I_B^{(i)}(t)$ between the interval midpoints.
- Since all process information is used in the estimates, satisfactory curve estimates can be obtained for a much higher value of $\Delta$ than the one needed for the pointwise estimates.

# Estimates of asymptotic availability and importance

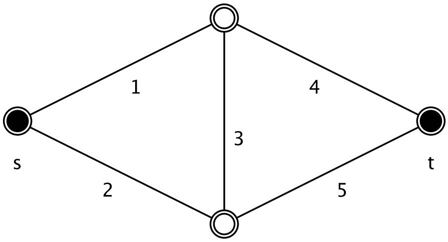
We may also obtain estimates of the asymptotic availability and importance by calculating averages over the intervals $(0, t_j]$, $j = 1, 2, \ldots$:

$$\bar{A}_\phi(t_j) = \frac{1}{M} \sum_{s=1}^{M} \frac{1}{t_j} \sum_{k \in \mathcal{F}_{sj}} \phi(\boldsymbol{X}_s(T_s^{(k)}))(T_s^{(k+1)} - T_s^{(k)}),$$

$$\bar{I}_B^{(i)}(t_j) = \frac{1}{M} \sum_{s=1}^{M} \frac{1}{t_j} \sum_{k \in \mathcal{F}_{sj}} \psi_i(\boldsymbol{X}_s(T_s^{(k)}))(T_s^{(k+1)} - T_s^{(k)}),$$

where $\mathcal{F}_{sj}$ denotes the index set of the events in $(0, t_j]$ in the $s$th simulation.

# Example: A bridge system




The five components in the system have exponential lifetime and repair time distributions with mean values 1 time unit.

**Objective**: Estimate $A_\phi(t)$ and $I_B^{(1)}(t), \ldots, I_B^{(5)}(t)$ for $t \in [0, 1000]$.

# Stationary values

Since in this *very particular case* all the lifetimes and repair times are exponentially distributed with the *same mean*, component availabilities can easily be calculated analytically:

$$N_i(t) = \text{Number of failure/repair events affecting comp. } i \text{ in } [0, t].$$

Now, we note that:

- $N_i(t)$ has a Poisson distribution with mean $t$
- $X_i(t) = 1$ if and only if $N_i(t)$ is even

Hence:

$$A_i(t) = \sum_{k=0}^{\infty} \Pr(N_i(t) = 2k) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} e^{-t}.$$

# Stationary values (cont.)

Convergence of the system availability:

$$|A_\phi(t) - A_\phi| < 10^{-15}, \text{ for } t > 20$$

Convergence of the Birnbaum measures of importance:

$$|I_B^{(i)}(t) - I_B^{(i)}| < 10^{-15}, \text{ for } t > 20, \; i = 1, \ldots, 5.$$

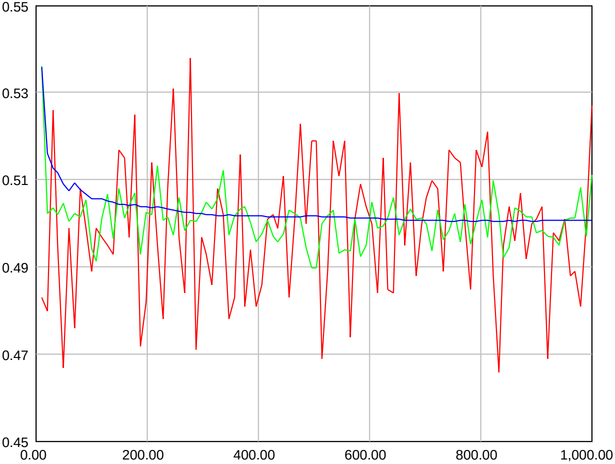



Figure: Availability curve estimates, $\hat{A}_\phi(t)$ (red curve), $\tilde{A}_\phi(t)$ (green curve) and $\bar{A}_\phi(t)$ (blue curve), $M = 1000$ simulations, $N = 100$ sample points, $\Delta = 10$ units

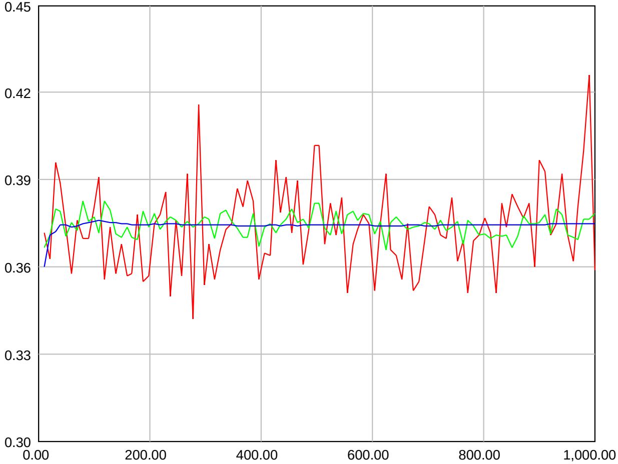



Figure: Importance curve estimates, $\hat{I}_B^{(1)}(t)$ (red curve) and $\tilde{I}_B^{(1)}(t)$ (green curve) and $\bar{I}_B^{(1)}(t)$ (blue curve), $M = 1000$ simulations, $N = 100$ sample points, $\Delta = 10$ units