STK3405 – Exercises Chapter 3

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Exercise 3.1

Prove equation (3.2) in another way than what is done in the proof of Theorem 3.1.1.

We need to show that the reliability function of a monotone system where the component state variables are independent, satisfies the following:

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p}), \quad i = 1, 2, \ldots, n.$$

PROOF: By conditioning on the state of component i we have:

$$\begin{aligned} h(\boldsymbol{p}) &= E[\phi(\boldsymbol{X})] \\ &= E[\phi(\boldsymbol{X}) | X_i = 1] P(X_i = 1) + E[\phi(\boldsymbol{X}) | X_i = 0] P(X_i = 0) \\ &= p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p}). \end{aligned}$$

Exercise 3.2

Prove Theorem 3.2.5:

Theorem

Let (C, ϕ) be a binary monotone system, and let (C^D, ϕ^D) be its dual. Then the following statements hold:

- $\boldsymbol{x}$ *is a path vector (alternatively, cut vector) for* (C, ϕ) *if and only if* $\boldsymbol{x}^D$ *is a cut vector (path vector) for* (C^D, ϕ^D).
- *A minimal path set (alternatively, cut set) for* (C, ϕ) *is a minimal cut set (path set) for* (C^D, ϕ^D).

Exercise 3.2 (cont.)

PROOF: Assume that $\boldsymbol{x}$ is a *path vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 1$ and we get:

$$\phi^D(\boldsymbol{x}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{x}^D) = 1 - \phi(\boldsymbol{x}) = 1 - 1 = 0.$$

Hence, $\boldsymbol{x}^D$ is a *cut vector* for (C^D, ϕ^D).

Similarly, assume that $\boldsymbol{x}$ is a *cut vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 0$ and we get:

$$\phi^D(\boldsymbol{x}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{x}^D) = 1 - \phi(\boldsymbol{x}) = 1 - 0 = 1.$$

Hence, $\boldsymbol{x}^D$ is a *path vector* for (C^D, ϕ^D).

Exercise 3.2 (cont.)

Assume that $P \subseteq C$ is a *minimal path set* for (C, ϕ), and let $\boldsymbol{x}_1$ be the corresponding *minimal path vector*. That is, $\boldsymbol{x}_1 = (\mathbf{1}^P, \mathbf{0})$.

Then by the first part of the theorem $\boldsymbol{x}_1^D = \mathbf{1} - \boldsymbol{x}_1$ is a cut vector for (C^D, ϕ^D).

Now, let $\boldsymbol{y}^D > \boldsymbol{x}_1^D$. This implies that:

$$\boldsymbol{y} = \mathbf{1} - \boldsymbol{y}^D < \mathbf{1} - \boldsymbol{x}_1^D = \boldsymbol{x}_1$$

Since $\boldsymbol{x}_1$ is a minimal path vector for (C, ϕ), $\boldsymbol{y}$ must be a cut vector for (C, ϕ).

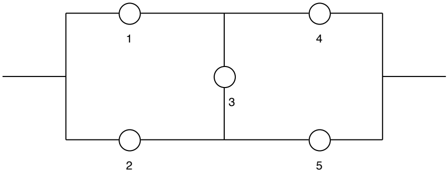
Hence, by the first part of the theorem $\boldsymbol{y}^D$ is a path vector for (C^D, ϕ^D), and since this holds for any $\boldsymbol{y}^D > \boldsymbol{x}_1^D$, we conclude that $\boldsymbol{x}_1^D$ is a minimal cut vector for (C^D, ϕ^D), implying that:

$$K = \{i : x_{i1}^D = 0\} = \{i : x_{i1} = 1\} = P$$

is a minimal cut set for (C^D, ϕ^D).

Exercise 3.3

Find all of the path and cut sets of the brigde structure:



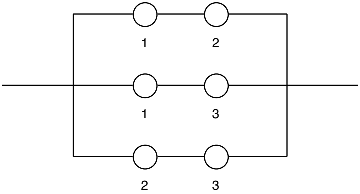
Minimal path sets:
$P_1 = \{1, 4\}$, $P_2 = \{1, 3, 5\}$, $P_3 = \{2, 3, 4\}$, $P_4 = \{2, 5\}$.

Minimal cut sets:
$K_1 = \{1, 2\}$, $K_2 = \{1, 3, 5\}$, $K_3 = \{2, 3, 4\}$, $K_4 = \{4, 5\}$.

Exercise 3.4

Find the representations via the minimal path sets and the minimal cut sets:

(i) **2-out-3 system:**



Minimal path sets: $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$, $P_3 = \{2, 3\}$.

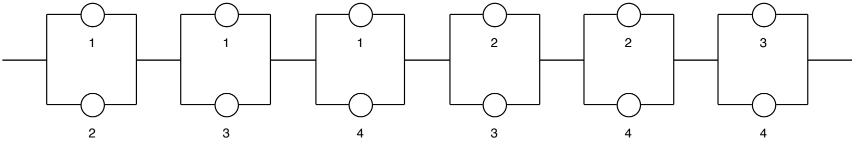
$\phi(\boldsymbol{X}) = (X_1 \cdot X_2) \amalg (X_1 \cdot X_3) \amalg (X_2 \cdot X_3)$

Minimal cut sets: $K_1 = \{1, 2\}$, $K_2 = \{1, 3\}$, $K_3 = \{2, 3\}$.

$\phi(\boldsymbol{X}) = (X_1 \amalg X_2) \cdot (X_1 \amalg X_3) \cdot (X_2 \amalg X_3)$

Exercise 3.4 (cont.)

(ii) **3-out-4 system:**



Minimal path sets:

$P_1 = \{1, 2, 3\}$, $P_2 = \{1, 2, 4\}$, $P_3 = \{1, 3, 4\}$, $P_4 = \{2, 3, 4\}$.

$$\phi(\boldsymbol{X}) = (X_1 \cdot X_2 \cdot X_3) \amalg (X_1 \cdot X_2 \cdot X_4) \amalg (X_1 \cdot X_3 \cdot X_4) \amalg (X_2 \cdot X_3 \cdot X_4)$$

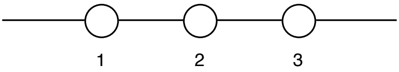
Minimal cut sets:

$K_1 = \{1, 2\}$, $K_2 = \{1, 3\}$, $K_3 = \{1, 4\}$, $K_4 = \{2, 3\}$, $K_5 = \{2, 4\}$, $K_6 = \{3, 4\}$.

$$\phi(\boldsymbol{X}) = (X_1 \amalg X_2) \cdot (X_1 \amalg X_3) \cdot (X_1 \amalg X_4) \cdot (X_2 \amalg X_3) \cdot (X_2 \amalg X_4) \cdot (X_3 \amalg X_4)$$

Exercise 3.4 (cont.)

(iii) **Series system of 3 components:**



Minimal path set: $P = \{1, 2, 3\}$.

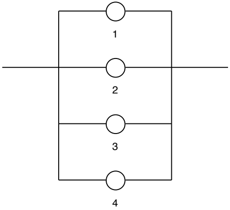
$\phi(\boldsymbol{X}) = X_1 \cdot X_2 \cdot X_3$

Minimal cut sets: $K_1 = \{1\}$, $K_2 = \{2\}$, $K_3 = \{3\}$.

$\phi(\boldsymbol{X}) = X_1 \cdot X_2 \cdot X_3$

Exercise 3.4 (cont.)

(iv) **Parallel system of 4 components:**



Minimal path set: $P_1 = \{1\}$, $P_2 = \{2\}$, $P_3 = \{3\}$, $P_4 = \{4\}$.

$\phi(\boldsymbol{X}) = X_1 \amalg X_2 \amalg X_3 \amalg X_4$

Minimal cut sets: $K = \{1, 2, 3, 4\}$.

$\phi(\boldsymbol{X}) = X_1 \amalg X_2 \amalg X_3 \amalg X_4$

Exercise 3.5

Consider a coherent system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$. Prove that

$$\bigcup_{j=1}^{p} P_j = \bigcup_{j=1}^{k} K_j = C.$$

NOTE: Since all the minimal path and cut sets are subsets of the component set C, we obviously have that:

$$\bigcup_{j=1}^{p} P_j \subseteq C \text{ and } \bigcup_{j=1}^{k} K_j \subseteq C$$

In order to show that for coherent systems the three sets are equal we show a slightly more general result.

Exercise 3.5 (cont.)

Theorem

Let (C, ϕ) be a binary monotone system. Then the following three statements are equivalent:

- *$i \in C$ is relevant*
- *$i \in P$ for at least one minimal path set P*
- *$i \in K$ for at least one minimal cut set K*

PROOF: Assume that $i \in C$ is relevant. Then there exists a vector $(\cdot_i, \boldsymbol{x})$ such that:

$$\phi(1_i, \boldsymbol{x}) = 1 \quad \text{and} \quad \phi(0_i, \boldsymbol{x}) = 0 \qquad (*)$$

NOTE: We can always choose $(\cdot_i, \boldsymbol{x})$ such that it is a *minimal* vector with the property $(*)$ in the sense that if $(\cdot_i, \boldsymbol{y}) < (\cdot_i, \boldsymbol{x})$, then $\phi(1_i, \boldsymbol{y}) = \phi(0_i, \boldsymbol{y}) = 0$.

Exercise 3.5 (cont.)

It then follows that $(1_i, \boldsymbol{x})$ is a minimal path vector since $\boldsymbol{y} < (1_i, \boldsymbol{x})$ implies that $\phi(\boldsymbol{y}) = 0$.

We then let $P = \{j \in C : x_j = 1\}$, which by definition this is a minimal path set. Moreover, $i \in P$.

Thus, we have shown that if $i \in C$ is relevant, then $i \in P$ for at least one minimal path set P.

We now show the converse implication, i.e., that if $i \in P$ for at least one minimal path set P, then $i \in C$ is relevant.

Exercise 3.5 (cont.)

Assume that there exists a minimal path set P such that $i \in P$, and let $\boldsymbol{x}$ be the corresponding minimal path vector. Then by definition:

$$x_i = 1 \quad \text{and} \quad \phi(\boldsymbol{x}) = \phi(1_i, \boldsymbol{x}) = 1.$$

Moreover, if $\boldsymbol{y} < \boldsymbol{x}$, then $\phi(\boldsymbol{y}) = 0$. Since in particular $(0_i, \boldsymbol{x}) < \boldsymbol{x}$, it follows that $\phi(0_i, \boldsymbol{x}) = 0$, i.e., i is relevant.

Thus, we have shown that if $i \in P$ for at least one minimal path set P, then $i \in C$ is relevant.

The proof that a component $i \in C$ is relevant if and only if $i \in K$ for at least one minimal cut set K is proved in a similar way.

Exercise 3.5 (cont.)

As direct consequence of the theorem we just proved we get:

Corollary

Consider a coherent system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$. We then have:

$$\bigcup_{j=1}^{p} P_j = \bigcup_{j=1}^{k} K_j = C.$$

PROOF: If (C, ϕ) is coherent, then *all* components are relevant. Hence, every component is contained in at least one minimal path set and in at least one minimal cut set.

Hence, the three sets must be equal.