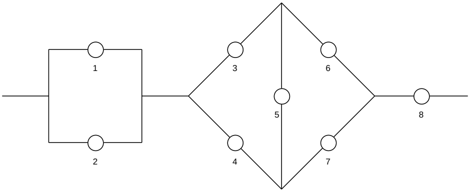
STK3405 – Exercises Chapter 3

A. B. Huseby & K. R. Dahl

Department of Mathematics
University of Oslo, Norway

Exercise 3.6

Find the minimal path sets and the minimal cut sets of the system in the system below.



Minimal path sets:

$P_1 = \{1, 3, 6, 8\}$, $P_2 = \{1, 3, 5, 7, 8\}$, $P_3 = \{1, 4, 5, 6, 8\}$, $P_4 = \{1, 4, 7, 8\}$, $P_5 = \{2, 3, 6, 8\}$, $P_6 = \{2, 3, 5, 7, 8\}$, $P_7 = \{2, 4, 5, 6, 8\}$, $P_8 = \{2, 4, 7, 8\}$.

Minimal cut sets:

$K_1 = \{1, 2\}$, $K_2 = \{3, 4\}$, $K_3 = \{3, 5, 7\}$, $K_4 = \{4, 5, 6\}$, $K_5 = \{6, 7\}$, $K_6 = \{8\}$.

Exercise 3.6 (cont.)

Find two different expressions for the structure function of this system.

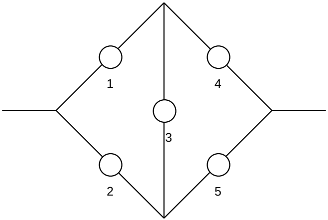
$$\phi(\boldsymbol{X}) = \coprod_{j=1}^{8} \prod_{i \in P_j} X_i$$

$$\phi(\boldsymbol{X}) = \prod_{j=1}^{6} \coprod_{i \in K_j} X_i$$

$$\phi(\boldsymbol{X}) = (X_1 \amalg X_2) \cdot [X_5 \cdot (X_3 \amalg X_4) \cdot (X_6 \amalg X_7) + (1 - X_5) \cdot ((X_3 X_6) \amalg (X_4 X_7))] \cdot X_8$$

Exercise 3.7

Show that if (C, ϕ) is a bridge structure, then (C^D, ϕ^D) is a bridge structure as well.



Minimal path sets:

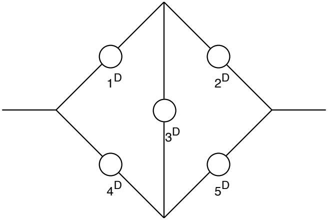
$P_1 = \{1, 4\}$, $P_2 = \{1, 3, 5\}$, $P_3 = \{2, 3, 4\}$, $P_4 = \{2, 5\}$

Minimal cut sets:

$K_1 = \{1, 2\}$, $K_2 = \{1, 3, 5\}$, $K_3 = \{2, 3, 4\}$, $K_4 = \{4, 5\}$.

Exercise 3.7 (cont.)

The dual system (C^D, ϕ^D):



Minimal path sets:
$P_1 = \{1^D, 2^D\}$, $P_2 = \{1^D, 3^D, 5^D\}$, $P_3 = \{2^D, 3^D, 4^D\}$, $P_4 = \{4^D, 5^D\}$

Minimal cut sets:
$K_1 = \{1^D, 4^D\}$, $K_2 = \{1^D, 3^D, 5^D\}$, $K_3 = \{2^D, 3^D, 4^D\}$, $K_4 = \{2^D, 5^D\}$.

Exercise 3.8

Let (A, χ) be a module of (C, ϕ). Assume that $\boldsymbol{x}_1$ and $\boldsymbol{x}_0$ are such that $\chi(\boldsymbol{x}_1^A) = 1$ and $\chi(\boldsymbol{x}_0^A) = 0$. Prove that for all $(\cdot^A, \boldsymbol{x}^{\bar{A}})$ we have:

$$\phi(\boldsymbol{x}_1^A, \boldsymbol{x}_1^{\bar{A}}) = \phi(\mathbf{1}^A, \boldsymbol{x}_1^{\bar{A}}) \text{ and } \phi(\boldsymbol{x}_0^A, \boldsymbol{x}_0^{\bar{A}}) = \phi(\mathbf{0}^A, \boldsymbol{x}_1^{\bar{A}}).$$

PROOF: Let ψ be the organising structure function for ϕ and χ. That is, we have:

$$\phi(\boldsymbol{x}) = \psi(\chi(\boldsymbol{x}^A), \boldsymbol{x}^{\bar{A}}) \quad \text{for all } \boldsymbol{x}.$$

Hence, since χ is non-decreasing we have by the assumptions that:

$$\begin{aligned}\phi(\boldsymbol{x}_1) &= \psi(\chi(\boldsymbol{x}_1^A), \boldsymbol{x}_1^{\bar{A}}) = \psi(1, \boldsymbol{x}_1^{\bar{A}}) \\ &= \psi(\chi(\mathbf{1}^A), \boldsymbol{x}_1^{\bar{A}}) = \phi(\mathbf{1}^A, \boldsymbol{x}_1^{\bar{A}}).\end{aligned}$$

Similarly:

$$\begin{aligned}\phi(\boldsymbol{x}_0) &= \psi(\chi(\boldsymbol{x}_0^A), \boldsymbol{x}_0^{\bar{A}}) = \psi(0, \boldsymbol{x}_0^{\bar{A}}) \\ &= \psi(\chi(\mathbf{0}^A), \boldsymbol{x}_0^{\bar{A}}) = \phi(\mathbf{0}^A, \boldsymbol{x}_0^{\bar{A}}).\end{aligned}$$

Exercise 3.9

Find all the modules of the following structure function:

$$\phi(\boldsymbol{x}) = (x_1 \cdot (x_2 \amalg x_3)) \amalg (x_4 \amalg x_5).$$

SOLUTION: (Only non-trivial modules are included here)

- $A_1 = \{2,3\}, \quad \chi_1(\boldsymbol{x}^{A_1}) = x_2 \amalg x_3, \quad \psi_1(\chi_1, \boldsymbol{x}^{\bar{A}_1}) = (x_1 \cdot \chi_1) \amalg (x_4 \amalg x_5)$
- $A_2 = \{1,2,3\}, \quad \chi_2(\boldsymbol{x}^{A_2}) = x_1(x_2 \amalg x_3), \quad \psi_2(\chi_2, \boldsymbol{x}^{\bar{A}_2}) = \chi_2 \amalg (x_4 \amalg x_5)$
- $A_3 = \{4,5\}, \quad \chi_3(\boldsymbol{x}^{A_3}) = x_4 \amalg x_5, \quad \psi_3(\chi_3, \boldsymbol{x}^{\bar{A}_3}) = (x_1(x_2 \amalg x_3)) \amalg \chi_3$
- $A_4 = \{1,2,3,4\}, \quad \chi_4(\boldsymbol{x}^{A_4}) = (x_1(x_2 \amalg x_3)) \amalg x_4, \quad \psi_4(\chi_4, \boldsymbol{x}^{\bar{A}_4}) = \chi_4 \amalg x_5$
- $A_4 = \{1,2,3,5\}, \quad \chi_5(\boldsymbol{x}^{A_5}) = (x_1(x_2 \amalg x_3)) \amalg x_5, \quad \psi_5(\chi_5, \boldsymbol{x}^{\bar{A}_5}) = \chi_5 \amalg x_4$

Exercise 3.10

Let (C, ϕ) be a k-out-of-n system where $1 < k < n$, and assume that (A, χ) is a module of (C, ϕ) such that $1 < |A| < n$.

We can then find a minimal path set P (i.e., a set where $|P| = k$) such that:

$$A \setminus P \neq \emptyset \quad \text{and} \quad A \cap P \neq \emptyset \quad \text{and} \quad P \setminus A \neq \emptyset.$$

It then follows that:

$$\phi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{P \setminus A}, \mathbf{0}) = 1,$$
$$\phi(\mathbf{0}^{A \setminus P}, \mathbf{0}^{A \cap P}, \mathbf{1}^{P \setminus A}, \mathbf{0}) = 0.$$

If $\chi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}) = 0$, by Exercise 3.8 this implies that:

$$1 = \phi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{P \setminus A}, \mathbf{0}) = \phi(\mathbf{0}^{A \setminus P}, \mathbf{0}^{A \cap P}, \mathbf{1}^{P \setminus A}, \mathbf{0}) = 0.$$

That is, we have arrived at a contradiction.

Exercise 3.10 (cont.)

Since $(P \setminus A) \neq \emptyset$ and $A \setminus P \neq \emptyset$, we can find a component $i \in (P \setminus A)$ and a component $j \in (A \setminus P)$.

Since $|(A \cap P) \cup ((P \setminus A) \setminus i)| = |P \setminus i| = k - 1$, we have:

$$\phi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{(P \setminus A) \setminus i}, \mathbf{0}) = 0.$$

Since $|(A \setminus P) \cup (A \cap P) \cup ((P \setminus A) \setminus i)| \geq |j \cup (P \setminus i)| = k$, we have:

$$\phi(\mathbf{1}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{(P \setminus A) \setminus i}, \mathbf{0}) = 1.$$

If $\chi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}) = 1$, by Exercise 3.8 this implies that:

$$0 = \phi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{(P \setminus A) \setminus i}, \mathbf{0}) = \phi(\mathbf{1}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{(P \setminus A) \setminus i}, \mathbf{0}) = 1.$$

That is, we have arrived at a contradiction.

Exercise 3.10 (cont.)

Since both $\chi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}) = 0$ and $\chi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}) = 1$ lead to contradictions, we conclude that it is not possible to find any binary function $\chi(\boldsymbol{x}^A)$ such that (A, χ) is a module of (C, ϕ).

Hence, A cannot be a modular set of (C, ϕ).

Since this is true for all sets $A \subseteq C$ such that $1 < |A| < n$, we conclude that a k-out-of-n system (C, ϕ) where $1 < k < n$ has *no* non-trivial modules.