STK3405 - Exercise 3.11-3.15

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Exercise 3.11

Consider the system shown in the figure below.

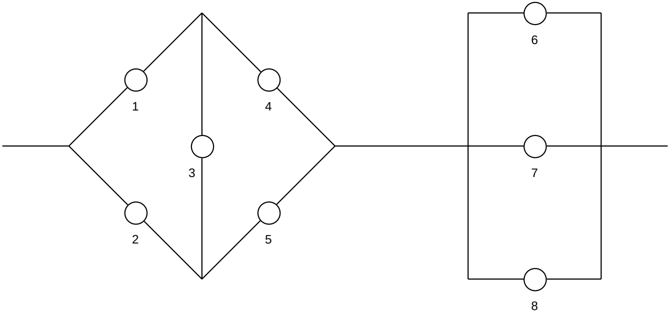


Figure: A system suited for modular decomposition.

Exercise 3.11 (cont.)

a) Find the structure function of the system by dividing the system into modules.

SOLUTION: We introduce the modules (A_1, χ_1) and (A_2, χ_2) where:

$$A_1 = \{1, 2, 3, 4, 5\}$$

$$\chi_1(\mathbf{x}^{A_1}) = x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))$$

$$A_2 = \{6, 7, 8\}$$

$$\chi_2(\mathbf{x}^{A_2}) = x_6 \amalg x_7 \amalg x_8$$

Since the two modules are connected in series, we get that:

$$\phi(\mathbf{x}) = \chi_1(\mathbf{x}^{A_1}) \cdot \chi_2(\mathbf{x}^{A_2})$$

$$= [x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))] \cdot [x_6 \amalg x_7 \amalg x_8]$$

Exercise 3.11 (cont.)

b) Find the structure function of the system without dividing the system into modules.

SOLUTION: There are several ways of doing this. Here, we choose to pivot w.r.t. component 3 (i.e., the brigde):

$$\phi(\mathbf{x}) = x_3\phi(1_3, \mathbf{x}) + (1 - x_3)\phi(0_3, \mathbf{x})$$

Case 1. Component 3 is *functioning*.
In this case the first system becomes a series of two parallel systems consisting of $1, 2$ and $4, 5$. This is again in series with the parallel connection of $6, 7, 8$. Hence we get:

$$\phi(1_3, \mathbf{x}) = (x_1 \amalg x_2) \cdot (x_4 \amalg x_5) \cdot (x_6 \amalg x_7 \amalg x_8)$$

Exercise 3.11 (cont.)

Case 2. Component 3 is *not functioning*.
In this case the first system becomes a parallel connection of the series connection of $1, 4$ and the series connection of $2, 5$. This is again is series with the parallel connection of $6, 7, 8$. Hence we get:

$$\phi(0_3, \mathbf{x}) = \big((x_1 x_4) \amalg (x_2 x_5)\big) \cdot (x_6 \amalg x_7 \amalg x_8)$$

Combining the two cases we get:

$$\begin{aligned}\phi(\mathbf{x}) = {} & x_3(x_1 \amalg x_2) \cdot (x_4 \amalg x_5) \cdot (x_6 \amalg x_7 \amalg x_8) \\ & + (1 - x_3)\big((x_1 x_4) \amalg (x_2 x_5)\big) \cdot (x_6 \amalg x_7 \amalg x_8).\end{aligned}$$

Exercise 3.11 (cont.)

c) Compare the computational efforts in a) and b).

SOLUTION: In a) $\phi(\mathbf{x})$ is expressed as:

$$\phi(\mathbf{x}) = [x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))] \cdot [x_6 \amalg x_7 \amalg x_8]$$

In b) $\phi(\mathbf{x})$ is expressed as:

$$\begin{aligned}\phi(\mathbf{x}) = {} & x_3(x_1 \amalg x_2) \cdot (x_4 \amalg x_5) \cdot (x_6 \amalg x_7 \amalg x_8) \\ & + (1 - x_3)\big((x_1 x_4) \amalg (x_2 x_5)\big) \cdot (x_6 \amalg x_7 \amalg x_8).\end{aligned}$$

The formula obtained via the modular decomposition is easier to compute than the formula obtained with a direct approach since with the latter approach we need to compute $(x_6 \amalg x_7 \amalg x_8)$ twice.

Exercise 3.12

Consider the system shown in the figure below.

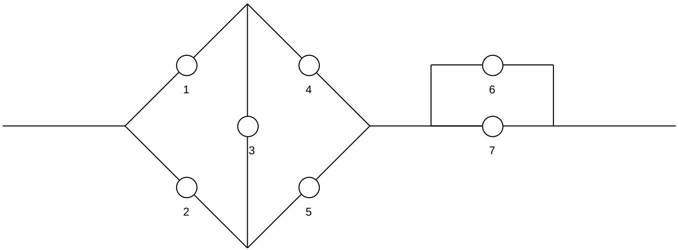


Figure: A mixed bridge and parallel system.

a) What is the structure function of this system?

Exercise 3.12 (cont.)

SOLUTION: We see that this is a bridge structure in series with a parallel structure.

Hence, by modular decomposition the structure function becomes:

$$\phi(\mathbf{x}) = \chi_1(\mathbf{x}^{A_1}) \cdot \chi_2(\mathbf{x}^{A_2})$$

where:

$$A_1 = \{1, 2, 3, 4, 5\}$$

$$\chi_1(\mathbf{x}^{A_1}) = x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))$$

$$A_2 = \{6, 7\}$$

$$\chi_2(\mathbf{x}^{A_2}) = x_6 \amalg x_7$$

Exercise 3.12 (cont.)

b) Assume that all of the component states are independent, and that $P(X_i = 1) = p_i$, $i = 1, \ldots, 7$. What is the reliability function of the system?

SOLUTION: Using the independence we get:

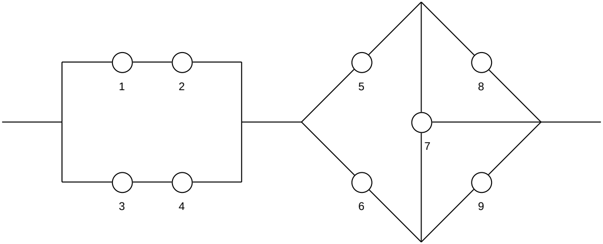
$$h(\mathbf{p}) = E[\phi(\mathbf{X})]$$

$$= E\big[[X_3(X_1 \amalg X_2)(X_4 \amalg X_5) + (1 - X_3)((X_1 X_4) \amalg (X_2 X_5))][X_6 \amalg X_7]\big]$$

$$= \big[E[X_3] \cdot E[X_1 \amalg X_2] E[X_4 \amalg X_5] + (1 - E[X_3]) \cdot E[(X_1 X_4) \amalg (X_2 X_5)]\big] \cdot E[X_6 \amalg X_7]$$

$$= \big[p_3 \cdot (p_1 \amalg p_2)(p_4 \amalg p_5) + (1 - p_3) \cdot ((p_1 p_4) \amalg (p_2 p_5))\big] \cdot (p_6 \amalg p_7)$$

Exercise 3.13



NOTE: In this undirected network system component 7 is *not an edge*. Here component 7 should be interpreted as a *node*. This must be taken into account when we do a pivotal decomposition with respect to component 7.

Exercise 3.13 (cont.)

a) What is the structure function of this system?

SOLUTION: We start out by decomposing the system into two modules (A_1, χ_1) and (A_2, χ_2) where $A_1 = \{1, 2, 3, 4\}$ and $A_2 = \{5, 6, 7, 8, 9\}$.

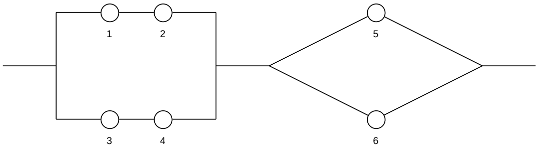
Since the first module is a parallel connection of the series connection of $1, 2$ and the series connection $3, 4$, we get:

$$\chi_1(\mathbf{x}^{A_1}) = (x_1 x_2) \amalg (x_3 x_4)$$

To find $\chi_2(\mathbf{x}^{A_2})$, we do a pivotal decomposition w.r.t. component 7:

$$\chi_2(\mathbf{x}^{A_2}) = x_7 \chi_2(1_7, \mathbf{x}^{A_2}) + (1 - x_7)\chi_2(0_7, \mathbf{x}^{A_2})$$

Exercise 3.13 (cont.)

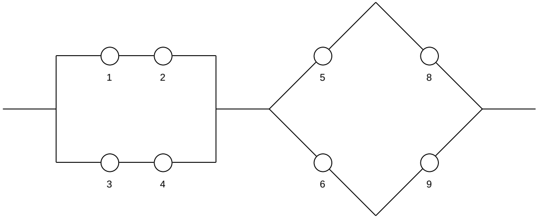


Case 1. Component 7 is *functioning*.

In this case components 8 and 9 become irrelevant and this module reduces to a parallel of components 5 and 6:

$$\chi_2(1_7, \mathbf{x}^{A_2}) = x_5 \amalg x_6$$

Exercise 3.13 (cont.)



Case 2. Component 7 is *not functioning*.

In this case the module becomes a parallel connection of the series connection of 5 and 8 and the series connection of 6 and 9:

$$\chi_2(0_7, \mathbf{x}^{A_2}) = ((x_5 x_8) \amalg (x_6 x_9))$$

Exercise 3.13 (cont.)

Combining the two cases we get:

$$\chi_2(\mathbf{x}^{A_2}) = x_7\chi_2(1_7, \mathbf{x}^{A_2}) + (1 - x_7)\chi_2(0_7, \mathbf{x}^{A_2})$$
$$= x_7(x_5 \amalg x_6) + (1 - x_7)((x_5 x_8) \amalg (x_6 x_9))$$

Exercise 3.13 (cont.)

Finally, since the two modules are in series we get:

$$\phi(\mathbf{x}) = \chi_1(\mathbf{x}^{A_1}) \cdot \chi_2(\mathbf{x}^{A_2})$$

$$= \big[(x_1 x_2) \amalg (x_3 x_4)\big] \cdot \big[x_7(x_5 \amalg x_6) + (1 - x_7)((x_5 x_8) \amalg (x_6 x_9))\big]$$

b) Assume that all of the component states are independent. What is the reliability function of the system?

SOLUTION: By a similar kind of calculation as in Exercise 3.12 b), using the independence assumption repeatedly, we find that:

$$h(\mathbf{p}) = E[\chi_1(\mathbf{x}^{A_1})] \cdot E[\chi_2(\mathbf{x}^{A_2})]$$

$$= E\big[(X_1 X_2) \amalg (X_3 X_4)\big] \cdot E\big[X_7(X_5 \amalg X_6) + (1 - X_7)((X_5 X_8) \amalg (X_6 X_9))\big]$$

$$= \big[(p_1 p_2) \amalg (p_3 p_4)\big] \cdot \big[p_7(p_5 \amalg p_6) + (1 - p_7)((p_5 p_8) \amalg (p_6 p_9))\big]$$

Exercise 3.14

Consider a binary monotone system (C, ϕ) with minimal cut sets $K_1, \ldots, K_k$, where T_i denotes the lifetime of the ith component, $i = 1, \ldots, n$, and where S denotes the lifetime of the system. Prove that:

$$S = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i$$

SOLUTION: Let U_j denote the liftetime of the jth minimal cut parallel structure. Then U_j equals the lifetime of the longest living component in K_j. That is, we have:

$$U_j = \max_{i \in K_j} T_i$$

Moreover, the lifetime of the system equals the lifetime of the shortest living minimal cut parallel structure. That is, we have:

$$S = \min_{1 \leq j \leq k} U_j = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i$$

Exercise 3.15

Consider a binary monotone system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$ and let f be a non-negative function defined over the positive integers $\{1, \ldots, n\}$. Prove that

$$\min_{1 \leq j \leq k} \max_{i \in K_j} f(i) = \max_{1 \leq j \leq p} \min_{i \in P_j} f(i).$$

SOLUTION: Since f takes inputs in $C = \{1, \ldots, n\}$ and is non-negative, $f(i)$ can be interpreted as the lifetime of the ith component, i.e., T_i. By Theorem 3.4.1 we know that:

$$\max_{1 \leq j \leq p} \min_{i \in P_j} T_i = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i$$

Hence, by replacing T_i by $f(i)$ for all $i \in C$, we get:

$$\max_{1 \leq j \leq p} \min_{i \in P_j} f(i) = \min_{1 \leq j \leq k} \max_{i \in K_j} f(i)$$

which was what we wanted to prove.