STK3405 - Exercise 4.1-4.5

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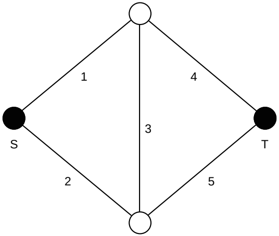
Exercise 4.1

Consider the bridge structure with independent component state variables and reliability function $h(\mathbf{p})$, and assume that $p_i = 0.9$, $i = 1, \ldots, 5$.



What do the bounds:

$$1 - \sum_{j=1}^{k} P(F_j) \leq h \leq \sum_{j=1}^{p} P(E_j)$$

reduce to in this case? Comment on the result.

Exercise 4.1 (cont.)

SOLUTION: The minimal path and minimal cut sets are:

$$P_1 = \{1, 4\}, \quad P_2 = \{1, 3, 5\}, \quad P_3 = \{2, 3, 4\}, \quad P_4 = \{2, 5\}$$

$$K_1 = \{1, 2\}, \quad K_2 = \{1, 3, 5\}, \quad K_3 = \{2, 3, 4\}, \quad K_4 = \{4, 5\}$$

Moreover, the reliability function is:

$$h(\mathbf{p}) = p_3(p_1 \amalg p_2)(p_4 \amalg p_5) + (1 - p_3)((p_1 p_4) \amalg (p_2 p_5)).$$

We then introduce:

$$E_j = \text{All the components in } P_j \text{ are functioning}, \quad j = 1, 2, 3, 4$$

$$F_j = \text{All the components in } K_j \text{ are failed}, \quad j = 1, 2, 3, 4$$

Exercise 4.1 (cont.)

Since $p_i = 0.9$, $i = 1, 2, 3, 4, 5$, we get:

$$P(E_1) = P(E_4) = 0.9^2, \quad P(E_2) = P(E_3) = 0.9^3$$

$$P(F_1) = P(F_4) = 0.1^2, \quad P(F_2) = P(F_3) = 0.1^3$$

Hence, the lower bound is given by:

$$1 - \sum_{j=1}^{4} P(F_j) = 1 - 2 \cdot 0.1^2 - 2 \cdot 0.1^3 = 0.978$$

and the upper bound is given by:

$$\sum_{j=1}^{4} P(E_j) = 2 \cdot 0.9^2 + 2 \cdot 0.9^3 = 3.078$$

Exercise 4.1 (cont.)

Finally, the exact reliability of the system is given by:

$$\begin{aligned} h(\mathbf{p}) &= p_3(p_1 \amalg p_2)(p_4 \amalg p_5) + (1 - p_3)((p_1 p_4) \amalg (p_2 p_5)) \\ &= 0.9 \cdot (0.9 \amalg 0.9)^2 + 0.1 \cdot (0.9^2 \amalg 0.9^2) \\ &= 0.9 \cdot (1 - (1 - 0.9)^2)^2 + 0.1 \cdot (1 - (1 - 0.9^2)^2) \\ &= 0.9 \cdot 0.99^2 + 0.1 \cdot (1 - 0.19^2) \\ &= 0.9 \cdot 0.9801 + 0.1 \cdot 0.9639 = 0.88209 + 0.09639 = 0.97848 \end{aligned}$$

Summarizing this we get the bounds:

$$0.978 \leq 0.97848 \leq 3.078$$

The lower bound is very good, while the upper bound is very bad (even greater than 1). According to the compendium this holds in general when the component reliabilities are close to 1.

Exercise 4.2

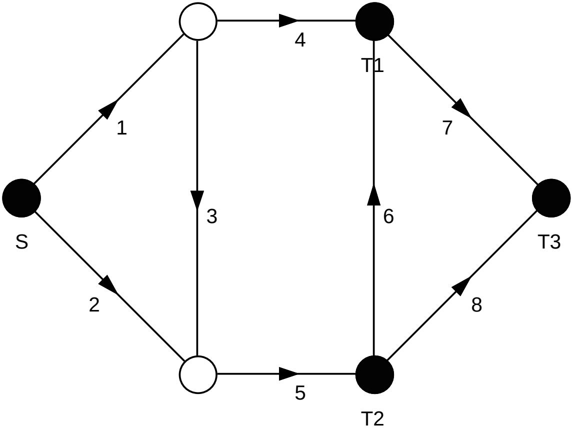
Draw an example of the following systems:

- An S3T system.
- An S5T system.

NOTE: This exercise can of course be solved in many different ways. The systems illustrated here are just some examples.

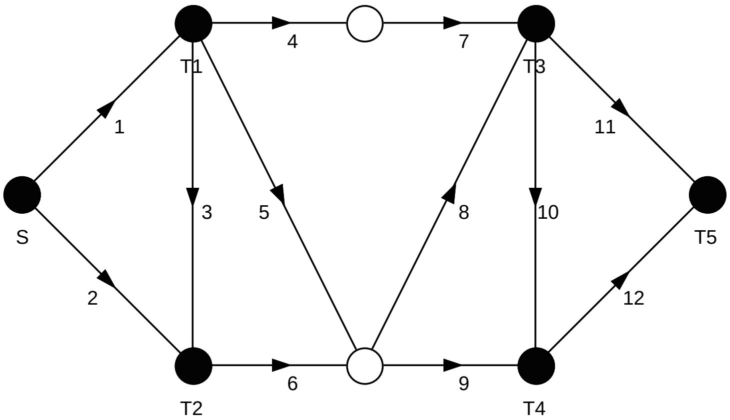
Exercise 4.2 (cont.)

An S3T system:



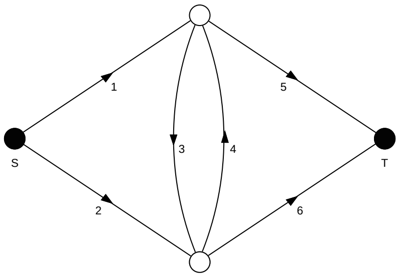
Exercise 4.2 (cont.)

An S5T system:



Exercise 4.3

Consider the S1T system below.



a) Find the minimal path sets of the system.

b) How many terms will there be in the inclusion-exclusion formula for the reliability of this system before simplification?

c) How many of these terms will vanish in the final simplified expression?

Exercise 4.3 (cont.)

SOLUTION:

a) The minimal path sets are:

$$P_1 = \{1,5\}, P_2 = \{2,6\}, P_3 = \{2,4,5\}, P_4 = \{1,3,6\}$$

b) The number of terms in the inclusion-exclusion formula based on the minimal path sets are (before cancelling terms):

$$2^p - 1 = 2^4 - 1 = 15.$$

Exercise 4.3 (cont.)

c)

$$h = P(\bigcup_{j=1}^{4} E_j)$$

$$\begin{aligned}
&= P(E_1) + P(E_2) + P(E_3) + P(E_4) \\
&- P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_1 \cap E_4) - P(E_2 \cap E_3) \\
&- P(E_2 \cap E_4) - P(E_3 \cap E_4) \\
&+ P(E_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_4) + P(E_1 \cap E_3 \cap E_4) \\
&+ P(E_2 \cap E_3 \cap E_4) \\
&- P(E_1 \cap E_2 \cap E_3 \cap E_4).
\end{aligned}$$

By checking which component set each of the event sets above corresponds to, we see that 4 out of the 15 terms are cancelled because they correspond to the same component set, but have opposite signs.

NOTE: The terms that cancel are the three last ones, as well as $-P(E_3 \cap E_4)$. They all correspond to the set of components $\{1, 2, 3, 4, 5, 6\}$.

Exercise 4.4

A linear consecutive k-out-of-n system is a binary monotone system (C, ϕ) with $P \subseteq C$ is a minimal path set if and only if P can be written as:

$$P = \{i, (i+1), \ldots, (i+k-1)\},$$

where $1 \leq k \leq n$, and

$$1 \leq i < i+k-1 \leq n.$$

Let (C, ϕ) be a linear consecutive 2-out-of-5 system. Show that (C, ϕ) cannot be an SKT-system.

[*Hint:* Let δ denote the signed domination function of the system. Compare $\delta(\{1, 2, 3, 4\})$ and $\delta(\{1, 2, 3, 4, 5\})$ and interpret the result in the light of Theorem 4.4.3.]

Exercise 4.4 (cont.)

SOLUTION: By definition a linear consecutive k-out-of-n system it follows that the minimal path sets of (C, ϕ) are:

$$P_1 = \{1, 2\}, \quad P_2 = \{2, 3\}, \quad P_3 = \{3, 4\}, \quad P_4 = \{4, 5\}$$

From this we find the structure function:

$$\begin{aligned}
\phi(\mathbf{x}) &= 1 - (1 - x_1 x_2)(1 - x_2 x_3)(1 - x_3 x_4)(1 - x_4 x_5) \\
&= x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 \\
&\quad - x_1 x_2 x_3 - x_1 x_2 x_3 x_4 - x_1 x_2 x_4 x_5 - x_2 x_3 x_4 - x_2 x_3 x_4 x_5 - x_3 x_4 x_5 \\
&\quad + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_2 x_3 x_4 x_5 \\
&\quad - x_1 x_2 x_3 x_4 x_5 \\
&= x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 \\
&\quad - x_1 x_2 x_3 - x_2 x_3 x_4 - x_3 x_4 x_5 - x_1 x_2 x_4 x_5 \\
&\quad + x_1 x_2 x_3 x_4 x_5
\end{aligned}$$

Exercise 4.4 (cont.)

We now consider the structure function:

$$\begin{aligned}\phi(\mathbf{x}) = & \; x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 \\ & - x_1 x_2 x_3 - x_2 x_3 x_4 - x_3 x_4 x_5 - x_1 x_2 x_4 x_5 \\ & + x_1 x_2 x_3 x_4 x_5\end{aligned}$$

We observe that $\delta(\{1, 2, 3, 4\}) = 0$ since the term $x_1 x_2 x_3 x_4$ does not occur in $\phi(\mathbf{x})$.

Moreover, we see that $\delta(\{1, 2, 3, 4, 5\}) = 1$

Exercise 4.4 (cont.)

If (C, ϕ) is an SKT-system, it follows from Theorem 4.4.3 that we must have:

$$\delta(A) = \begin{cases} (-1)^{|A|-v(A)+1} & \text{if } A \text{ is a union of minimal path sets} \\ & \text{and not containing a circuit} \\ 0 & \text{otherwise} \end{cases}$$

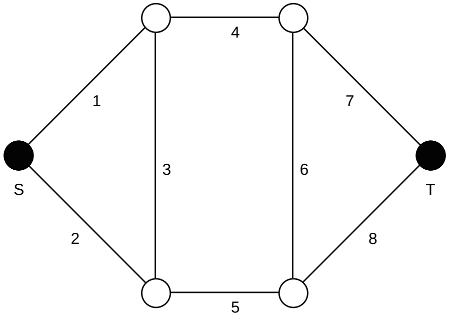
We then note that $\{1, 2, 3, 4\} = P_1 \cup P_3$, while $\delta(\{1, 2, 3, 4\}) = 0$. The only way this could happen if (C, ϕ) is an SKT-system, is if the set $\{1, 2, 3, 4\}$ contains a circuit.

However, this implies that the larger set $\{1, 2, 3, 4, 5\}$ must contain a circuit as well, which is impossible since $\delta(\{1, 2, 3, 4, 5\}) = 1$.

Hence, we conclude that (C, ϕ) cannot be an SKT-system.

Exercise 4.5

Compute the reliability function of the undirected network system below which functions if and only if the nodes S and T can communicate through the network. Use the factoring algorithm.



Exercise 4.5 (cont.)

SOLUTION: We do a pivotal decomposition w.r.t. component 3:

$$h(\mathbf{p}) = p_3 h_{+3}(\mathbf{p}) + (1 - p_3) h_{-3}(\mathbf{p})$$

For the system $(C \setminus 3, \phi_{+3})$ we get a parallel of 1 and 2 in series with a bridge structure. Hence, we get:

$$h_{+3}(\mathbf{p}) = (p_1 \amalg p_2) \cdot \big(p_6(p_4 \amalg p_5)(p_7 \amalg p_8) + (1 - p_6)(p_4 p_7 \amalg p_5 p_8)\big).$$

For the system $(C \setminus 3, \phi_{-3})$ we get a bridge structure, but where the first component is replaces by a series of 1 and 4 and the second component is replaced by a series of 2 and 5. Hence, we get:

$$h_{-3}(\mathbf{p}) = p_6((p_1 p_4) \amalg (p_2 p_5)) \cdot (p_7 \amalg p_8) + (1 - p_6)((p_1 p_4 p_7) \amalg (p_2 p_5 p_8)).$$