STK3405 - Exercise 6.3-6.6

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Exercise 6.3: Prove the upper bound of Corollary 6.2.8.

SOLUTION: The upper bound is

$$h(\mathbf{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i.$$

To prove this, note that for independent component state variables, the reliability of the minimal path series structures is

$$P(\rho_j(\mathbf{X}^{P_j}) = 1) = E[\prod_{i \in P_j} X_i] = \prod_{i \in P_j} p_i.$$

Hence,

$$h(\mathbf{p}) \leq \coprod_{j=1}^{p} P(\rho_j(\mathbf{X}^{P_j}) = 1) = \coprod_{j=1}^{p} \prod_{i \in P_j} p_i$$

where the inequality follows from Corollary 6.2.6 (or alternatively, by using the same proof technique as in this result).

Exercise 6.4: Prove the upper bound in Corollary 6.2.6 by applying the lower bound on the dual structure function ϕ^D.

SOLUTION: We know that

$$\max_{1 \leq j \leq p} \prod_{i \in P_j} p_i \leq h.$$

Applied to the dual structure function ϕ^D, we get:

$$\max_{1 \leq j \leq p^D} \prod_{i \in P_j^D} p_i^D \leq h^D.$$

From the definition of h^D, p_i^D and the fact that minimal path sets for ϕ are minimal cut sets for ϕ^D, this is equivalent with

$$\max_{1 \leq j \leq k} \prod_{i \in K_j} (1 - p_i) \leq 1 - h.$$

Hence, using that $\max\{\cdot\} = -\min\{-\cdot\}$ and the definition of the coproduct,

$$
\begin{aligned}
h &\leq 1 - \max_{1 \leq j \leq k} \prod_{i \in K_j} (1 - p_i) \\
&= \min_{1 \leq j \leq k} (1 - \prod_{i \in K_j} (1 - p_i)) \\
&= \min_{1 \leq j \leq k} \coprod_{i \in K_j} p_i
\end{aligned}
$$

which is the upper bound of Corollary 6.2.6.

Exercise 6.5: Prove the upper bound in Corollary 6.2.8 by applying the lower bound on the dual structure function ϕ^D.

SOLUTION: We know that

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\mathbf{p}).$$

Applied to the dual structure,

$$\prod_{j=1}^{k^D} \coprod_{i \in K_j^D} p_i^D \leq h^D(\mathbf{p}^D).$$

By using the definition of h^D, p_i^D and that minimal path sets of ϕ are minimal cut sets of ϕ^D, we find that this is equivalent to

$$\prod_{j=1}^{p} \coprod_{i \in P_j} (1 - p_i) \leq 1 - h(\mathbf{p}).$$

That is,

$$\begin{aligned} h(\mathbf{p}) &\leq 1 - \prod_{j=1}^{p} \coprod_{i \in P_j} (1 - p_i) \\ &= 1 - \prod_{j=1}^{p} (1 - \prod_{i \in P_j} (1 - (1 - p_j))) \\ &= 1 - \prod_{j=1}^{p} (1 - \prod_{i \in p_j} p_i) \\ &= \coprod_{j=1}^{p} \prod_{i \in P_j} p_i \end{aligned}$$

where we have used the definition of the coproduct and some algebra. Note that this is the upper bound we wanted to prove, so this concludes our solution.

Exercise 6.6: Consider the network in Figure 1 consisting of independent components with component reliabilities *p*.

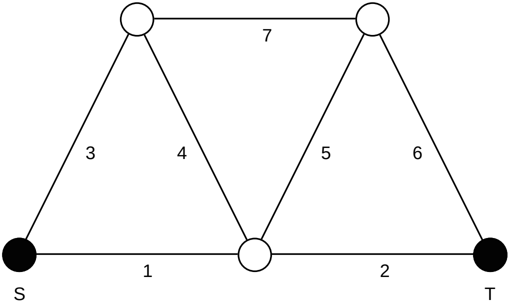

Figure: Illustration of the network for Exercise 6.6.

a) What is the reliability $h(p)$ for this system?

SOLUTION: We factor w.r.t. component 7:

$$h(\mathbf{p}) = ph(1_7, \mathbf{p}) + (1 - p)h(0_7, \mathbf{p}).$$

If component 7 works (make a drawing!): In this case, components 4 and 5 are in parallel. Hence, we can parallel reduce these two components to a new component 4' where $p_{4'} = p \coprod p$.
The resulting structure is a bridge structure, so

$$\begin{aligned} h(1_7, \mathbf{p}) &= p_{4'}(p \coprod p)(p \coprod p) + (1 - p_{4'})(p^2 \coprod p^2) \\ &= \dots \\ &= (p \coprod p)^3 + (1 - p)^2(p^2 \coprod p^2). \end{aligned}$$

If component 7 doesn't work (make a drawing!): The resulting system is an s-p system. In this case, components 3 and 4 are in series, and so are 5 and 6. Hence, we can series reduce 3 and 4 to a new component 3' with reliability $p_{3'} = p^2$. Similarly, we can series reduce components 5 and 6 to a new component $5'$ with reliability $p_{5'} = p^2$. Then, we can parallel reduce 3' and 1 to a new component 1' with reliability $p_{1'} = p \coprod p^2$. Similarly, we can parallel reduce 5' and 2 to a new component 2' with reliability $p_{2'}$. Then,

$$
\begin{aligned}
h(0_7, \mathbf{p}) &= p_{1'} p_{2'} \\
&= (p \coprod p^2)^2.
\end{aligned}
$$

So,

$$h(\mathbf{p}) = p\{(p \coprod p)^3 + (1-p^2)(p^2 \coprod p^2)\} + (1-p)(p \coprod p^2)^2.$$

b) For $p \in [0, 1]$, write a program to compute the reliability $h(p)$. Plot $h(p)$.

SOLUTION:

Please ask via e-mail if you have trouble with this. You should end up with an S-shaped curve starting at (0,0) and ending in (1,1).

c) In the same plot, illustrate the bounds from Corollary 6.2.6 and 6.2.8. Comment on the result.

SOLUTION:

Recall from Corollary 6.2.6 that

$$\max_{1 \le j \le p} \prod_{i \in P_j} p_i \le h \le \min_{1 \le j \le k} \coprod_{i \in K_j} p_i.$$

The minimal path and cuts sets are:

Minimal path sets: $P_1 = \{1,2\}$, $P_2 = \{2,3,4\}$, $P_3 = \{1,5,6\}$, $P_4 = \{3,4,5,6\}$, $P_5 = \{3,6,7\}$, $P_6 = \{2,3,5,7\}$, $P_7 = \{1,4,6,7\}$.

Minimal cut sets: $K_1 = \{1,3\}$, $K_2 = \{2,6\}$, $K_3 = \{1,4,7\}$, $K_4 = \{2,3,4,5\}$, $K_5 = \{2,5,7\}$, $K_6 = \{1,4,5,6\}$.

So the lower bound is

$$\max_{1 \leq j \leq p} \prod_{i \in P_j} p = \max\{p^2, p^3, p^4\} = p^2$$

since $p \in [0, 1]$.

Similarly, the upper bound is

$$\min_{1 \leq j \leq k} \coprod_{i \in K_j} p = \min\{1-(1-p)^2, 1-(1-p)^3, 1-(1-p)^4\} = 1-(1-p)^2$$

since $p \in [0, 1]$.

From Corollary 6.2.8,

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\mathbf{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i.$$

Hence, the lower bound is

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p = (1-(1-p)^2)^2(1-(1-p)^3)^2(1-(1-p)^4)^2.$$

And the upper bound is

$$\coprod_{j=1}^{p} \prod_{i \in P_j} p = 1-(1-p^2)(1-p^3)^3(1-p^4)^3.$$

If you plot all of these bounds, you see that the bounds from Corollary 6.2.8 look overall better than those from Corollary 6.2.6. However, they are not always better: For small p's, the lower bound from 6.2.6 is better than that from 6.2.8. For large p's, the upper bound from 6.2.6 is better than that from 6.2.8.

d) Is it possible to improve these bounds further?

SOLUTION:

To improve the bounds further, one can take the maximum of the two lower bounds and similarly a minimum of the two upper bounds. This will still be lower and upper bounds, respectively, but these new bounds will be better than those from Corollary 6.2.6 and Corollary 6.2.8.