# STK3405 - Exercise 5.1-5.6

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**Exercise 5.1:** Let $(C, \phi)$ be a $k$-out-of-$n$ system. Prove that all the components of this system have the same Birnbaum measure for structural importance.

**SOLUTION:** Let $i \in C$ be a component in the system. Then,

$$\begin{aligned} J_B^{(i)} &= \tfrac{1}{2^{n-1}} \sum_{(\cdot_i, \mathbf{x})} (\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})) \\ \phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) &= \begin{cases} 1 & \text{if } k-1 \text{ other components function} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So we sum over all $\mathbf{x} \in \mathbb{R}^{n-1}$ such that $k-1$ components function.

The number of ways to choose e.g., $k-1$ out of $n-1$ is

$$\frac{(n-1)!}{(k-1)!(n-1-k+1)!} = \frac{(n-1)!}{(k-1)!(n-k)!}$$

So,

$$J_B^{(i)} = \frac{1}{2^{n-1}} \frac{(n-1)!}{(k-1)!(n-k)!} \text{ for all } i \in C$$

Note that there is no $i$-dependency, so all components have the same Birnbaum measure for structural importance.

**Exercise 5.2:** Compute $J_B^{(i)}$ for the components of the bridge structure and compare their structural importance.

**SOLUTION:**
**Structural importance of component 3:** Critical path vectors for component 3;

$$(1,0,1,0,1),(0,1,1,1,0); \text{ 2 critical path vectors.}$$

Hence,

$$J_B^{(3)} = \frac{2}{2^{5-1}} = \frac{1}{8}.$$

**Structural importance of remaining components:** For example component 1: Critical path vectors for component 1:

$$\begin{aligned}&(1,1,0,1,0),(1,0,1,1,0),(1,0,0,1,1)\\&(1,0,1,1,1),(1,0,1,0,1),(1,0,0,1,0); \text{ 6 critical path vectors.}\end{aligned}$$

Similar for components $2, 4$ and $5$ as well. Hence,

$$J_B^{(i)} = \frac{6}{2^{5-1}} = \frac{3}{8} \text{ for } i = 1, 2, 4, 5.$$

So components $1, 2, 4, 5$ have greater structural importance in the bridge structure than component 3 (the bridge).

**Exercise 5.3:** Let the $i$'th component be in series with the rest of a monotone structure $\phi$, while the $j$'th component is not. Prove that

$$J_B^{(i)} > J_B^{(j)}.$$

**SOLUTION:** Since $i$ is in series, all vectors must be critical path vectors for component $i$. This property only holds for components in series. So since $j$ is not in series, there must exist some vectors which are not critical path vectors for component $j$. Then, from the definition of the Birnbaum measure of structural importance,

$$J_B^{(i)} > J_B^{(j)}.$$

**Exercise 5.4:** Compute $I_B^{(i)}$ for the components of the bridge structure and compare their reliability importance.

**SOLUTION:** We know

$$\begin{aligned} h(\mathbf{p}) &= p_3(p_1 \amalg p_2)(p_4 \amalg p_5) + (1-p_3)((p_1p_4) \amalg (p_2p_5)) \\ &= p_3(p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5) \\ &\quad +(1-p_3)(p_1p_4 + p_2p_5 - p_1p_2p_4p_5). \end{aligned}$$

Hence,

$$I_B^{(1)} = \frac{\partial h(\mathbf{p})}{\partial p_1} = p_3(1-p_2)(p_4+p_5-p_4p_5) + (1-p_3)(p_4 - p_2p_4p_5).$$

$$I_B^{(2)} = \frac{\partial h(\mathbf{p})}{\partial p_2} = p_3(1-p_1)(p_4+p_5-p_4p_5) + (1-p_3)(p_4 - p_1p_4p_5).$$

$$I_B^{(3)} = \frac{\partial h(\mathbf{p})}{\partial p_3} = (p_1+p_2-p_1p_2)(p_4+p_5-p_4p_5) - (p_1p_4+p_2p_5-p_1p_2p_4p_5).$$

$$I_B^{(4)} = \frac{\partial h(\mathbf{p})}{\partial p_4} = p_3(p_1 + p_2 - p_1 p_2)(1 - p_5) + (1 - p_3)(p_1 - p_1 p_2 p_5).$$

$$I_B^{(5)} = \frac{\partial h(\mathbf{p})}{\partial p_5} = p_3(p_1 + p_2 - p_1 p_2)(1 - p_4) + (1 - p_3)(p_1 - p_1 p_2 p_4).$$

If, say, all components have the same reliability, so $p_i = p$, $i = 1, \ldots, 5$, then, for $i = 1, 2, 4, 5$:

$$\begin{aligned} I_B^{(i)} &= p(2p - p^2)(1 - p) + (1 - p)(p - p^3) \\ &= p(1 - p)(1 + 2p - 2p^2) \end{aligned}$$

and

$$\begin{aligned} I_B^{(3)} &= (2p - p^2)(2p - p^2) - (2p^2 - p^4) \\ &= \ldots \\ &= 2p^2(1 - p + p^2) \end{aligned}$$

When are these equal? Clearly if $p = 0$. If $p \neq 0$:

$$\begin{aligned} p(1-p)(1+2p-2p^2) &= 2p^2(1-p+p^2) \\ -2p^2 - p + 1 &= 0 \end{aligned}$$

Hence, $p = -1$ or $p = 0.5$. Since $p$ is a reliability, $p = -1$ is not possible. So, we are left with $p = 0.5$.

Inserting, say $p = 0.1$, we see that for $p \in [0, 0.5]$, $I_B^{(3)} \leq I_B^{(i)}$, $i = 1, 2, 4, 5$. For $p \in (0.5, 1]$, $I_B^{(3)} > I_B^{(i)}$, $i = 1, 2, 4, 5$.

Therefore, the reliability importance of component 3 is smaller for not-so-reliable components, but bigger for more reliable components (w.r.t. the Birnbaum measure).

**Exercise 5.5:** Assume that the component lifetimes have so-called *proportional hazards*, that is:

$$\bar{F}_i(t) = \exp(-\lambda_i R(t)), \quad \lambda_i > 0, t \geq 0, \quad i = 1, \ldots, n,$$

where $R$ is a strictly increasing, differentiable function such that $R(0) = 0$, and $\lim_{t\to\infty} R(t) = \infty$. Prove that for a series structure, we have:

$$I_{B-P}^{(i)} = I_N^{(i)} = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}.$$

**SOLUTION:** Recall that $\bar{F}_i := 1 - F_i$. Note that from the definition of $\bar{F}_i(t)$:

$$f_i(t) = -\frac{d\bar{F}_i(t)}{dt} = \lambda_i R'(t)\bar{F}_i(t)$$

and $I_B^{(i)}(t) = \prod_{j \neq i} \bar{F}_j(t) = \prod_{j \neq i} p_i(t)$ since we are considering a series structure.

Then,

$$
\begin{aligned}
I_{B-P}^{(i)} &= \int_0^\infty I_B^{(i)}(t) f_i(t) dt \\
&= \int_0^\infty \prod_{j=1}^n \bar{F}_j(t) R'(t) \lambda_i dt \\
&= \lambda_i \int_0^\infty e^{-\sum_{j=1}^n \lambda_j R(t)} R'(t) \lambda_i dt \\
&= \lambda_i [-\frac{1}{\sum_{j=1}^n \lambda_j} e^{-\sum_{j=1}^n \lambda_j R(t)}]_{t=0}^\infty \\
&= \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
\bar{F}_i(0) = 1, &\text{ so } R(0) = 0, \\
\bar{F}_i(\infty) = 0, &\text{ so } R(\infty) = \infty.
\end{aligned}
$$

For the Natvig measure,

$$\begin{aligned} E[Z_i] &= \int_0^\infty \bar{F}_i(t)(-\ln \bar{F}_i(t)) I_B^{(i)}(t)dt \\ &= \int_0^\infty \prod_{j=1}^n \bar{F}_j(t)\lambda_i R(t)dt \end{aligned}$$

So,

$$\begin{aligned} I_N^{(i)} &= \frac{E[Z_i]}{\sum_{k=1}^n E[Z_k]} \\ &= \frac{\lambda_i \int_0^\infty \prod \bar{F}_j(t)R(t)dt}{\sum_{k=1}^n \lambda_k \int_0^\infty \prod_{j=1}^n \bar{F}_j(t)R(t)dt} \\ &= \frac{\lambda_i}{\sum_{k=1}^n \lambda_k} \end{aligned}$$

So the Barlow Prochan and Natvig measures are the same in this case. Both rank the reliability importance of the components based on the size of the error rates $\lambda_i$. That is, the larger the error rate, the more important the component. This corresponds to the intuition that the poorest component is the most important in the series system.

**Exercise 5.6:** Assume that the *i*'th component is irrelevant for the system $\phi$. Then, what is $I_B^{(i)}(t)$, $I_{B-P}^{(i)}$ and $I_N^{(i)}$?

**SOLUTION:** By pivot decomposition,

$$
\begin{aligned}
I_B^{(i)}(t) &= \tfrac{dh(\mathbf{p}(t))}{dp_i(t)} \\
&= h(1_i, \mathbf{p}(t)) - h(0_i, \mathbf{p}(t)) \\
&= E[\phi(1_i, \mathbf{X}(t)) - \phi(0_i, \mathbf{X}(t))] \\
&= \textstyle\sum_{(\cdot_i, \mathbf{x})} (\phi(1_i, \mathbf{x}(t)) - \phi(0_i, \mathbf{x}(t))) P(\mathbf{X}(t) = \mathbf{x}) \\
&= 0
\end{aligned}
$$

since the $i$'th component is irrelevant.

Since,

$$\begin{aligned} I_{B-P}^{(i)} &= \int_0^\infty f_i(t) I_B^{(i)}(t)dt \\ E[Z_i] &= \int_0^\infty \bar{F}_i(t)(-\ln \bar{F}_i(t)) I_B^{(i)}(t)dt \\ I_N^{(i)} &= \frac{E[Z_i]}{\sum_{j=1}^n E[Z_j]}, \end{aligned}$$

it follows that

$$I_{B-P}^{(i)} = I_N^{(i)} = 0.$$

So according to these measures, the reliability importance of an irrelevant component is 0 (which is intuitive).