STK3405 - Case study: Transmission of electronic pulses

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Section 9.2: Case study - Reliability analysis of a network for transmission of electronic pulses

We will perform a reliability analysis of a system for transmission of electronic pulses.

We consider a *multistate* system: The system is not just functioning or failed, but the set of possible system states is a set of non-negative integers.

The structure function

Let:

$$X_i(t) = \text{ The state of component } i \text{ at time } t,$$

and assume that the stochastic processes $\{X_i(t), t \geq 0\}_{i=1}^n$ are independent. The structure function is given by:

$$\phi(t) = \phi(\mathbf{X}(t)) = \text{ The state of the system at time } t.$$

We assume that $\phi(t) \in \{\phi_1, \ldots, \phi_k\}$.

In principle, one can find the distribution of the state of a multistate system by enumerating all possible component states:

$$P(\phi(t) = \phi_j) = \sum_{\mathbf{x}} I(\phi(\mathbf{x}) = \phi_j) \cdot P(\mathbf{X}(t) = \mathbf{x}), \ j = 1, \ldots, k. \quad (1)$$

where the indicator function $I(\phi(\mathbf{x}) = \phi_j)$ is 1 if $\phi(\mathbf{x}) = \phi_j$ and 0 otherwise.

Reduce the number of terms

However, we typically need to reduce the number of terms because the calculations become too time-consuming.

Assume that there exists variables $Y_1 = Y_1(\mathbf{X}), \ldots, Y_m = Y_m(\mathbf{X})$ such that $\phi(t)$ can be written:

$$\phi(t) = \phi(\mathbf{X}(t)) = \phi(\mathbf{Y}(\mathbf{X}(t))),$$

where $\mathbf{Y}(\mathbf{X}(t)) = (Y_1(\mathbf{X}(t)), \ldots, Y_m(\mathbf{X}(t)))$.

Then, the probability distribution of ϕ can be found from the formula:

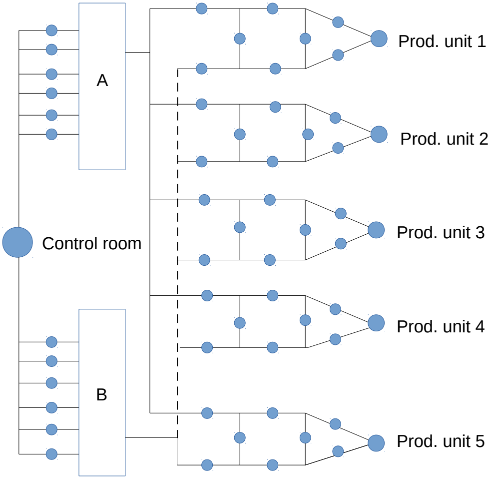
$$P(\phi(t) = \phi_j) = \sum_{\mathbf{y}} I(\phi(\mathbf{y}) = \phi_j) P(\mathbf{Y}(\mathbf{X}(t)) = \mathbf{y}), \; j = 1, \ldots, k. \tag{2}$$

We have to compute $P(\mathbf{Y}(\mathbf{X}(t)) = \mathbf{y})$ for all $\mathbf{y}$. Hence, if the number of possible values of $\mathbf{Y}$ is large, little is gained by using (2) instead of (1).

In some cases, we can achieve a large reduction in the computational load by a clever choice of $Y_1, \ldots, Y_m$.

The network for electronic pulses

In particular, this turns out to be the case for the following network:



The network

The purpose of the system shown in Figure 1 is to ensure communication between a control room and 5 production units.

In addition to these components, the system consists of a control room, 5 production units and two connection units called A and B.

Do not consider potential errors in the control room, the production units and the connection units. Hence, the system consists of $n = 52$ components.

The number of production units which can be controlled by a connection unit is bounded by the number of functioning input wires to the respective connection unit (if all 6 input wires to A are functioning, then all 5 production units can be controlled via A).

Identical subsystems

Assume that all of the components are stochastically independent.

Also assume that all of the 12 input wires to the connection units A and B have the same lifetime distributions and that for any wire between the control room and a connection unit we have:

$$P(\text{The wire is functioning at time } t) = w(t).$$

Note: System consist of 5 identical subsystems.

Each of these subsystems consists of 8 components. Assume that all of these subsystems have the same stochastic properties (same components have same lifetime distributions).

The state of the system

Define the state of the system, ϕ, as:

$$\phi(t) = \text{The number of production units which can be controlled from the control room at time } t.$$

The set of possible values for ϕ is $\{0, 1, \ldots, 5\}$.

The components are assumed to be either functioning or failed (the component states are binary).

The system in Figure 1 is a multi-state system of binary components.

The number of terms in the sum (1) (state enumeration) for computing the distribution of ϕ is $2^{52} = 4.504 \cdot 10^{15}$. Too time-consuming to compute!

Introduce auxiliary variables

By introducing appropriate variables, we can obtain a significant reduction of terms in (2) compared to (1).

Introduce:

$$
\begin{aligned}
Y_1(t) &= \text{The number of intact input wires to connection unit } A \text{ at time } t \\
Y_2(t) &= \text{The number of intact input wires to connection unit } B \text{ at time } t \\
Y_3(t) &= \text{The number of production units connected to } A \text{ and } B \text{ at time } t \\
Y_4(t) &= \text{The number of production units only connected to } A \text{ at time } t \\
Y_5(t) &= \text{The number of production units only connected to } B \text{ at time } t.
\end{aligned}
$$

System state expressed via the Ys

The state of the system can now be expressed as:

$$\phi(t) = W_1(t) + W_2(t) + W_3(t), \tag{3}$$

where:

$$W_1(t) = \min\{Y_1(t), Y_4(t)\}, \quad W_2(t) = \min\{Y_2(t), Y_5(t)\}, \tag{4}$$

$$W_3(t) = \min\{(Y_1(t) - W_1(t)) + (Y_2(t) - W_2(t)), Y_3(t)\}.$$

To derive equation (3): Distribute the $Y_1(t)$ functioning input wires to A and the $Y_2(t)$ functioning input wires to B between the 5 production units such that as many production units as possible are running.

Derivation of equation (3)

To do so, distribute input wires to the production units which are only connected to one connection unit: $Y_4(t)$ production units only connected to A and the $Y_5(t)$ production units only connected to B. Hence, $W_1(t)$ production units are connected via A and $W_2(t)$ connection units via B.

Now, there are $Y_1(t) - W_1(t)$ input wires available via A and $Y_2(t) - W_2(t)$ input wires available via B. Use these to establish connection to as many as possible of the $Y_3(t)$ remaining production units. The number of production units which can be reached in this way is then $W_3(t)$.

Note that both $Y_1(t) - W_1(t)$ an $Y_2(t) - W_2(t)$ may be 0.

Conclusion: $W_1(t) + W_2(t) + W_3(t)$ is the number of production units which can be controlled from the control room. Thus, equation (3) holds.

We need to find the probability distributions of $Y_1(t), \ldots, Y_5(t)$.

$Y_1(t)$ is a function of the state variables of the wires into A.

$Y_2(t)$ is a function of the state variables of the wires into unit B.

The vector $(Y_3(t), Y_4(t), Y_5(t))$ is a function of state variables of the other 40 components.

By assumption, the components are independent, so Y_1, Y_2 and the vector $(Y_3(t), Y_4(t), Y_5(t))$ are independent.

Note: $Y_3(t)$, $Y_4(t)$ and $Y_5(t)$ are dependent
$(0 \leq Y_3(t) + Y_4(t) + Y_5(t) \leq 5$ for all $t \geq 0)$.

The distribution of $Y_1(t)$ and $Y_2(t)$

Both $Y_1(t)$ and $Y_2(t)$ are sums of 6 independent, identically distributed binary random variables.

Hence, from standard probability theory, $Y_1(t)$ and $Y_2(t)$ are binomially distributed:

$$P(Y_i(t) = y) = \binom{6}{y}[w(t)]^y[1 - w(t)]^{6-y}, \quad y = 0, 1, \ldots, 6, \quad i = 1, 2, \tag{5}$$

The distribution of $(Y_3(t), Y_4(t), Y_5(t))$

Consider the probability distribution of $(Y_3(t), Y_4(t), Y_5(t))$:

Depends on the 5 subsystems for communication between the connection units and production units.

Each of these subsystems are in on of four states: S_{AB}, S_A, S_B and $S_\emptyset$, where:

$$
\begin{aligned}
S_{AB} &= \text{The prod. unit can communicate with both connection units} \\
S_A &= \text{The prod. unit can only communicate with connection unit } A \\
S_B &= \text{The prod. unit can only communicate with connection unit } B \\
S_\emptyset &= \text{The prod. unit cannot communicate with any connection unit.}
\end{aligned}
$$

Since, per assumption, the subsystems are independent with the same stochastic properties, the probability of being in a state S_{AB}, S_A, S_B or $S_\emptyset$ at time t is the same for all the subsystems.

At time $t, Y_3(t)$, $Y_4(t)$ and $Y_5(t)$ are the number of production units in states S_{AB}, S_A and S_B respectively.

From standard probability theory: For $t \geq 0$ the vector $(Y_3(t), Y_4(t), Y_5(t))$ is multinomially distributed:

$$P(Y_3(t) = y_3 \cap Y_4(t) = y_4 \cap Y_5(t) = y_5) = \frac{5!}{y_3!y_4!y_5!(5 - y_3 - y_4 - y_5)!} \tag{6}$$

$$\cdot [p_3(t)]^{y_3}[p_4(t)]^{y_4}[p_5(t)]^{y_5}[1 - p_3(t) - p_4(t) - p_5(t)]^{5-y_3-y_4-y_5}$$

where $y_3 = 0, 1, \ldots, 5$, $y_4 = 0, 1, \ldots, 5 - y_3$, $y_5 = 0, 1, \ldots, 5 - y_3 - y_4$ and $p_3(t), p_4(t), p_5(t)$ are the probabilities of a subsystem being in states S_{AB}, S_A and S_B respectively.

To compute the distribution of ϕ, all that remains is to find $p_3(t), p_4(t)$ and $p_5(t)$.

To do so, we study the structure of the subsystems:

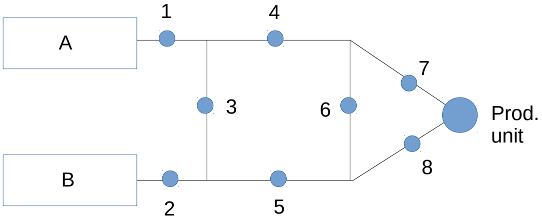


Figure: Subsystem of a network for transmission of electronic pulses.

In the figure, we number the subsystem components from 1-8.

Denote the corresponding component state variables by $X_1, \ldots, X_8$, and their respective reliabilities by $q_1, \ldots, q_8$ (omit the time t to simplify notation).

Introduce the events:

$$E_A = \{\text{The production unit communicates with connection unit } A\}$$
$$E_B = \{\text{The production unit communicates with connection unit } B\}.$$

Then,

$$\begin{aligned} p_3(t) &= P(E_A \cap E_B), \\ p_4(t) &= P(E_A) - P(E_A \cap E_B), \\ p_5(t) &= P(E_B) - P(E_A \cap E_B). \end{aligned} \tag{7}$$

Computing $P(E_A \cap E_B)$

Condition w.r.t. the two bridges in the structure, i.e., components 3 and 6:

$$\begin{aligned} P(E_A \cap E_B) = & P(E_A \cap E_B | X_3 = 1, X_6 = 1) q_3 q_6 \\ & + P(E_A \cap E_B | X_3 = 1, X_6 = 0) q_3 (1 - q_6) \\ & + P(E_A \cap E_B | X_3 = 0, X_6 = 1)(1 - q_3) q_6 \\ & + P(E_A \cap E_B | X_3 = 0, X_6 = 0)(1 - q_3)(1 - q_6). \end{aligned} \tag{8}$$

The four conditional probabilities in (8) can be computed by series- and parallel reductions (this is left as an exercise). They are given by:

$$\begin{aligned} P(E_A \cap E_B | X_3 = 1, X_6 = 1) &= q_1 q_2 [q_4 + q_5 - q_4 q_5][q_7 + q_8 - q_7 q_8], \\ P(E_A \cap E_B | X_3 = 1, X_6 = 0) &= q_1 q_2 [q_4 q_7 + q_5 q_8 - q_4 q_5 q_7 q_8], \\ P(E_A \cap E_B | X_3 = 0, X_6 = 1) &= q_1 q_2 q_4 q_5 [q_7 + q_8 - q_7 q_8], \\ P(E_A \cap E_B | X_3 = 0, X_6 = 0) &= q_1 q_2 q_4 q_5 q_7 q_8. \end{aligned}$$

Computing $P(E_A)$ and $P(E_B)$

To compute $P(E_A)$, component 2 is irrelevant (see the subsystem figure).

By removing this component, we see that components 3 and 5 are in series.

After a series reduction, we get a bridge structure connected in series with component 1. Hence, $P(E_A)$ is given by:

$$
\begin{aligned}
P(E_A) = & \; q_1 q_6 [q_4 + q_3 q_5 - q_3 q_4 q_5][q_7 + q_8 - q_7 q_8] \\
& + q_1 (1 - q_6)[q_4 q_7 + q_3 q_5 q_8 - q_3 q_4 q_5 q_7 q_8].
\end{aligned} \tag{9}
$$

Similarly, $P(E_B)$ is given by:

$$
\begin{aligned}
P(E_B) = & \; q_2 q_6 [q_5 + q_3 q_4 - q_3 q_4 q_5][q_7 + q_8 - q_7 q_8] \\
& + q_2 (1 - q_6)[q_5 q_8 + q_3 q_4 q_7 - q_3 q_4 q_5 q_7 q_8].
\end{aligned} \tag{10}
$$

Conclusion

By inserting the expressions (8), (9) and (10) into (7), we have all the probabilities we need in order to calculate the distribution of the system state ϕ using equation (2).

What is gained?

Had two alternatives:

1. State enumeration: Counting all possible states of the 52 binary state variables of the components in Figure 1.

2. Reduction by introducing auxiliary varibles: Have to enumerate the possible states of just the five multinary variables $Y_1, \ldots Y_5$.

To do this: Observe that Y_1 and Y_2 each attain 7 values in the set $\{0, 1, \ldots 6\}$.

The vector (Y_3, Y_4, Y_5) can attain any value in the set $\{(Y_3, Y_4, Y_5) : 0 \leq Y_3 + Y_4 + Y_5 \leq 5, 0 \leq Y_3, Y_4, Y_5\}$.

Counting the possible states of $Y_1, \ldots Y_5$

To count the number of values in this set, consider 6 different cases corresponding to the possible values of Y_3, and count the number of possible values for each case.

Note that if $Y_3 = y$, then $Y_4 + Y_5 \leq 5 - y$, $y = 0, 1, \ldots, 5$. Hence:

Case 0.	$Y_3 = 0,$	$Y_4 + Y_5 \leq 5 - 0 = 5:$	21 possible values,
Case 1.	$Y_3 = 1,$	$Y_4 + Y_5 \leq 5 - 1 = 4:$	15 possible values,
Case 2.	$Y_3 = 2,$	$Y_4 + Y_5 \leq 5 - 2 = 3:$	10 possible values,
Case 3.	$Y_3 = 3,$	$Y_4 + Y_5 \leq 5 - 3 = 2:$	6 possible values,
Case 4.	$Y_3 = 4,$	$Y_4 + Y_5 \leq 5 - 4 = 1:$	3 possible values,
Case 5.	$Y_3 = 5,$	$Y_4 + Y_5 \leq 5 - 5 = 0:$	1 possible value.

What is gained? 2744 vs. $4.504 \cdot 10^{15}$

Adding this, we see that there are 56 values in total.

Hence, the sum in (2) for computing $P(\phi(t) = \phi_j)$ contains $7 \cdot 7 \cdot 56 = 2744$ terms (since there are 7 possible values for Y_1 and Y_2, as well as 56 possible values for (Y_3, Y_4, Y_5)).

Easy to compute!

Alternative: The $4.504 \cdot 10^{15}$ terms of the original method by simply enumerating the states.

To conclude: By introducing the new variables $Y_1, \ldots, Y_5$, the computational load has been significantly reduced.

When to use this approach? Systems where the structure consists of many identical components and where there is a strong symmetry.

Difficulty with the method: In practice, we may not find efficient ways to introduce new variables.