STK3405 – Week 34 summary

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Overview

- System analysis:
  - Binary monotone systems
  - Coherent systems
  - Reliability of binary monotone systems
- Basic reliability calculation methods
  - Pivotal decompositions (conditioning)
  - Path and cut sets
  - Modules of monotone systems
- Exact computation of reliability
  - State space enumeration
  - The multiplication method
  - The inclusion-exclusion methods
  - Network reliability

Overview (cont.)

- Structural and reliability importance:
  - Structural importance of a component
  - Reliability importance of a component
  - Time-independent measures
- Association and reliability bounds
  - Associated random variables
  - Bounds on the system reliability
- Conditional Monte Carlo methods
  - Monte Carlo simulation and conditioning
  - Conditioning on the sum
  - Identical component reliabilities

Overview (cont.)

- Discrete event simulation:
  - Pure jump processes
  - Binary monotone systems of repairable components
  - Simulating repairable systems
  - Estimating availability and importance
- Applications
  - Case study: Fishing boat engines
  - Case study: Transmission of electronic pulses

Binary monotone systems

- A *system* is some technological unit consisting of a finite set of *components* which are operating together
- A *binary system* has only two possible states: *functioning* or *failed*. Moreover, each component is either functioning or failed as well
- A *binary monotone system* is a binary system such that repairing a component does not make the system worse, and breaking a component does not make system better

Binary monotone systems (cont.)

We consider a binary system of n components, $1, \ldots, n$, and introduce the component state variable x_i denoting the *state* of component i, $i = 1, \ldots, n$, defined as:

$$x_i = \begin{cases} 1 & \text{if the } i\text{th component is functioning} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we introduce the variable ϕ representing the *state* of the system, defined as:

$$\phi = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{otherwise.} \end{cases}$$

Binary monotone systems (cont.)

The variables x_i, $i = 1, \ldots, n$ and ϕ are said to be *binary*, since they only have two possible values, 0 and 1.

The state of the system is assumed to be uniquely determined by the state of the components. Thus, we may write ϕ as:

$$\phi = \phi(\boldsymbol{x}) = \phi(x_1, x_2, \ldots, x_n)$$

The function $\phi : \{0, 1\}^n \rightarrow \{0, 1\}$ is called the *structure function* of the system.

Binary monotone systems (cont.)

Definition

A binary monotone system is an ordered pair (C, ϕ), where:

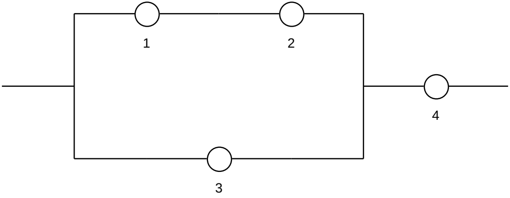
$C = \{1, \ldots, n\}$ is the component set and ϕ is the structure function.

The structure function ϕ is non-decreasing in each argument.

The number of components, n, is referred to as the *order* of the system.

Reliability block diagrams

In a *reliability block diagram* components are drawn as circles and connected by lines. The system is functioning if and only it is possible to find a way through the diagram passing only functioning components.



Note: In order to represent arbitrarily binary monotone systems, we allow components to occur in multiple places in the block diagram. Thus, a reliability block diagram should not necessarily be interpreted as a picture of a physical system.

A series system

Figure: A reliability block diagram of a series system.

A parallel system

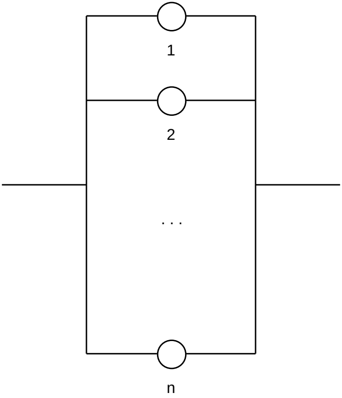


Figure: A reliability block diagram of a parallel system.

A 2-out-of-3 system

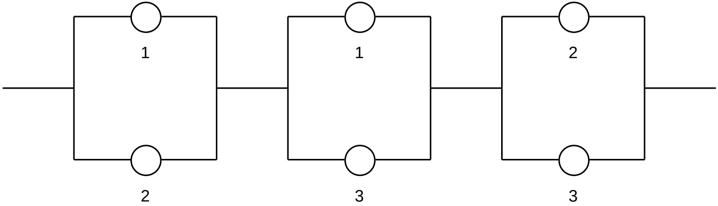


Figure: A reliability block diagram of a 2-out-of-3 system.

For a 2-out-of-3 system to fail 2 out of 3 components must fail. There are 3 possible subsets of components which contains 2 components: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$.

The structure function of a series system (cont.)

A series system is functioning if and only if all of its components are functioning, i.e., if and only if $x_1 = 1$ and $\cdots$ and $x_n = 1$.

Hence, the state of the system can be expressed as a function of the component state variables as follows:

$$\phi(\boldsymbol{x}) = x_1 \cdot x_2 \cdots x_n = \prod_{i=1}^{n} x_i$$

Alternatively, the structure function of a series system can be written as:

$$\phi(\boldsymbol{x}) = \min\{x_1, \ldots, x_n\}$$

since $\min\{x_1, \ldots, x_n\} = 1$ if and only if $x_1 = 1$ and $\cdots$ and $x_n = 1$.

The structure function of a parallel system (cont.)

A parallel system is functioning if and only if at least one of its components are functioning, i.e., if and only if $X_1 = 1$ or $\cdots$ or $X_n = 1$.

Hence, the state of the system can be expressed as a function of the component state variables as follows:

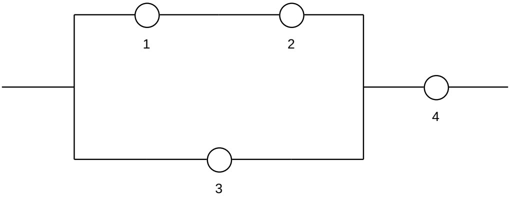
$$\phi(\boldsymbol{x}) = x_1 \amalg x_2 \cdots \amalg x_n = \coprod_{i=1}^{n} x_i.$$

Alternatively, the structure function of a parallel system can be written as:

$$\phi(\boldsymbol{x}) = \max\{x_1, \ldots, x_n\}$$

since $\max\{x_1, \ldots, x_n\} = 1$ if and only if $x_1 = 1$ or $\cdots$ or $x_n = 1$.

The structure function of a mixed system



It is easy to verify that the structure function of this system is:

$$\phi(\boldsymbol{x}) = [(x_1 \cdot x_2) \amalg x_3] \cdot x_4$$

Coherent systems

Every component of a binary monotone system should have some impact on the system state. More precisely, if i is a component in a system, there should ideally exist at least some state of the rest of the system where the system state depends on the state of component i.

Notation:

$$(1_i, \boldsymbol{x}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

$$(0_i, \boldsymbol{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

$$(\cdot_i, \boldsymbol{x}) = (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n).$$

Coherent systems (cont.)

Definition

Let (C, ϕ) be a binary monotone system, and let $i \in C$. The component i is said to be *relevant* for the system (C, ϕ) if:

$$0 = \phi(0_i, \boldsymbol{x}) < \phi(1_i, \boldsymbol{x}) = 1 \text{ for some } (\cdot_i, \boldsymbol{x}).$$

If this is not the case, component i is said to be *irrelevant* for the system.

A binary monotone system (C, ϕ) is *coherent* if all its components are relevant.

Note that a coherent system is obviously non-trivial as well, since in a trivial system *all* components are irrelevant.

The best and worst systems

Theorem

Let (C, ϕ) be a non-trivial binary monotone system of order n. Then for all $\boldsymbol{x} \in \{0, 1\}^n$ we have:

$$\prod_{i=1}^{n} x_i \leq \phi(\boldsymbol{x}) \leq \coprod_{i=1}^{n} x_i.$$

Component level changes vs. system level changes

Theorem

Let (C, ϕ) be a binary monotone system of order n. Then for all binary vectors $\boldsymbol{x}, \boldsymbol{y}$ we have:

(i) $\phi(\boldsymbol{x} \amalg \boldsymbol{y}) \geq \phi(\boldsymbol{x}) \amalg \phi(\boldsymbol{y})$,

(ii) $\phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y})$.

Moreover, assume that (C, ϕ) is coherent. Then equality holds in (i) for all $\boldsymbol{x}, \boldsymbol{y} \in \{0, 1\}^n$ if and only if (C, ϕ) is a parallel system. Similarly, equality holds in (ii) for all $\boldsymbol{x}, \boldsymbol{y} \in \{0, 1\}^n$ if and only if (C, ϕ) is a series system.

Interpretation: Components in parallel are better than systems in parallel. Components in series are worse than systems in series.

Dual systems

Definition

Let ϕ be a structure function of a binary monotone system of order n. We then define the *dual structure function*, ϕ^D for all $\boldsymbol{x} \in \{0, 1\}^n$ as:

$$\phi^D(\boldsymbol{x}) = 1 - \phi(\mathbf{1} - \boldsymbol{x}).$$

Furthermore, if $\boldsymbol{X}$ is the component state vector of a binary monotone system, we define the dual component state vector $\boldsymbol{X}^D$ as:

$$\boldsymbol{X}^D = (X_1^D, \ldots, X_n^D) = (1 - X_1, \ldots, 1 - X_n) = \mathbf{1} - \boldsymbol{X}$$

Dual systems (cont.)

Note:

- The relation between ϕ and ϕ^D is a relation between two *functions*
- The relation between $\boldsymbol{X}$ and $\boldsymbol{X}^D$ is a relation between two *stochastic vectors*

We also introduce the dual component set $C^D = \{1^D, \ldots, n^D\}$, where the dual component i^D is functioning if the component i is failed, while i^D is failed if the component i is functioning.

We have the following relation between the two stochastic variables $\phi(\boldsymbol{X})$ and $\phi^D(\boldsymbol{X}^D)$:

$$\phi^D(\boldsymbol{X}^D) = 1 - \phi(\boldsymbol{1} - \boldsymbol{X}^D) = 1 - \phi(\boldsymbol{X}).$$

Hence, the dual system is functioning if and only if the original system is failed and vice versa.

Examples of dual systems (cont.)

Let (C, ϕ) be a series system of order n:

$$\phi(\boldsymbol{x}) = \prod_{i=1}^{n} x_i.$$

The dual structure function is then given by:

$$\begin{aligned}\phi^D(\boldsymbol{x}) &= 1 - \phi(\mathbf{1} - \boldsymbol{x}) \\ &= 1 - \prod_{i=1}^{n}(1 - x_i) = \coprod_{i=1}^{n} x_i.\end{aligned}$$

Thus, (C^D, ϕ^D) is a parallel system of order n.

Examples of dual systems (cont.)

Let (C, ϕ) be a parallel system of order n:

$$\phi(\boldsymbol{x}) = \coprod_{i=1}^{n} x_i.$$

The dual structure function is then given by:

$$\begin{aligned} \phi^D(\boldsymbol{x}) &= 1 - \phi(\mathbf{1} - \boldsymbol{x}) = 1 - \coprod_{i=1}^{n}(1 - x_i) \\ &= 1 - (1 - \prod_{i=1}^{n}(1 - (1 - x_i)) = \prod_{i=1}^{n} x_i. \end{aligned}$$

Thus, (C^D, ϕ^D) is a series system of order n.

Dual systems (cont.)

Theorem

Let ϕ be the structure function of a binary monotone system, and let ϕ^D be the corresponding dual structure function. Then we have:

$$(\phi^D)^D = \phi.$$

That is, the dual of the dual system is equal to the original system.

Reliability of binary monotone systems

Let (C, ϕ) be a binary monotone system, and let $i \in C$.

$$p_i = P(X_i = 1) = \text{The } \textit{reliability} \text{ of a component } i$$

Since the state variable X_i is binary, we have for all $i \in C$:

$$\mathrm{E}[X_i] = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1) = p_i$$

Thus, the reliability of component i is equal to the expected value of its component state variable, X_i.

Reliability of binary monotone systems (cont.)

$$h = P(\phi(\boldsymbol{X}) = 1) = \text{The } \textit{reliability} \text{ of the system}$$

Since ϕ is binary, we have:

$$\mathrm{E}[\phi(\boldsymbol{X})] = 0 \cdot P(\phi(\boldsymbol{X}) = 0) + 1 \cdot P(\phi(\boldsymbol{X}) = 1) = P(\phi(\boldsymbol{X}) = 1) = h.$$

Thus, the reliability of the system is equal to the expected value of the structure function, $\phi(\boldsymbol{X})$.

From this it immediately follows that the reliability of a system, at least in principle, can be calculated as:

$$h = \mathrm{E}[\phi(\boldsymbol{X})] = \sum_{\boldsymbol{X} \in \{0,1\}^n} \phi(\boldsymbol{x}) P(\boldsymbol{X} = \boldsymbol{x})$$

Independent components

We now focus on the case where the component state variables can be assumed to be *independent* and introduce $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$. We note that:

$$P(X_i = x_i) = \begin{cases} p_i & \text{if } x_i = 1, \\ 1 - p_i & \text{if } x_i = 0. \end{cases}$$

Since x_i is either 0 or 1, $P(X_i = x_i)$ can be written in the following more compact form:

$$P(X_i = x_i) = p_i^{x_i}(1 - p_i)^{1 - x_i}.$$

The reliability function

Thus, when the component state variables are independent, their joint distribution can be written as:

$$P(\boldsymbol{X} = \boldsymbol{x}) = \prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}.$$

Hence, we get the following expression for the system reliability:

$$h = h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \sum_{\boldsymbol{x} \in \{0,1\}^n} \phi(\boldsymbol{x}) \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}$$

The function $h(\boldsymbol{p})$ is called *the reliability function* of the system.

Reliability of a series system

Consider a series system of order n. Assuming that the component state variables are independent, the reliability of this system is given by:

$$h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \mathrm{E}[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n} \mathrm{E}[X_i] = \prod_{i=1}^{n} p_i,$$

where the third equality follows since $X_1, X_2, \ldots, X_n$ are independent.

Reliability of a parallel system

Consider a parallel system of order n. Assuming that the component state variables are independent, the reliability of this system is given by:

$$h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \mathrm{E}[\coprod_{i=1}^{n} X_i] = \mathrm{E}[1 - \prod_{i=1}^{n}(1 - X_i)]$$
$$= 1 - \prod_{i=1}^{n}(1 - E[X_i]) = \coprod_{i=1}^{n} \mathrm{E}[X_i] = \coprod_{i=1}^{n} p_i,$$

where the fourth equality follows since $X_1, X_2, \ldots, X_n$ are independent.

Component level changes vs. system level changes

In the following we define $\boldsymbol{p} \cdot \boldsymbol{p}'$ as $(p_1 \cdot p_1', \ldots, p_n \cdot p_n')$.

Theorem

Let $h(\boldsymbol{p})$ be the reliability function of a binary monotone system (C, ϕ) of order n. Then for all $\boldsymbol{p}, \boldsymbol{p}' \in [0,1]^n$ we have:

(i) $h(\boldsymbol{p} \amalg \boldsymbol{p}') \geq h(\boldsymbol{p}) \amalg h(\boldsymbol{p}')$,

(ii) $h(\boldsymbol{p} \cdot \boldsymbol{p}') \leq h(\boldsymbol{p}) \cdot h(\boldsymbol{p}')$

If (C, ϕ) is coherent, equality holds in (i) for all $\boldsymbol{p}, \boldsymbol{p}' \in [0,1]^n$ if and only if (C, ϕ) is a parallel system.

If (C, ϕ) is coherent, equality holds in (ii) for all $\boldsymbol{p}, \boldsymbol{p}' \in [0,1]^n$ if and only if (C, ϕ) is a series system.