

# STK3405 – Week 35

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## Section 2.5

### ***k*-out-of-*n* systems**



## $k$ -out-of- $n$ systems

A  $k$ -out-of- $n$  system is a binary monotone system  $(C, \phi)$  where  $C = \{1, \dots, n\}$  which functions if and only if at least  $k$  out of the  $n$  components are functioning.

Let the component state variable of component  $i$  be  $X_i$ ,  $i \in C$ , and let the vector of component state variables be  $\mathbf{X} = (X_1, \dots, X_n)$ .

The structure function,  $\phi$ , can then be written:

$$\phi(\mathbf{X}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n X_i \geq k \\ 0 & \text{otherwise.} \end{cases}$$



# An $n$ -out-of- $n$ system = A series system

An  $n$ -out-of- $n$  system is the same as a series system:



**Figure:** A reliability block diagram of an  $n$ -out-of- $n$  system.



## A 1-out-of- $n$ system = A parallel system

A 1-out-of- $n$  system is the same as a parallel system:

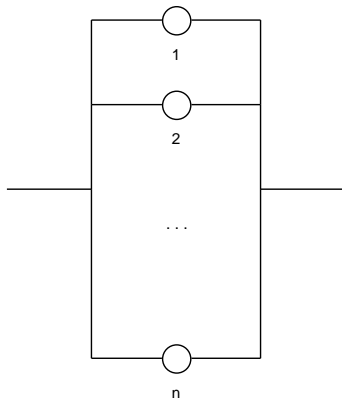
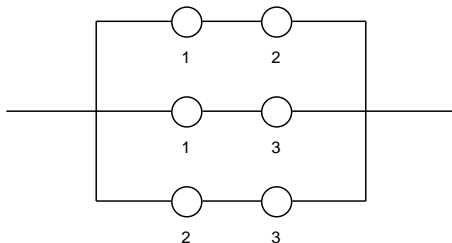


Figure: A reliability block diagram of an 1-out-of- $n$  system.



## A 2-out-of-3 system

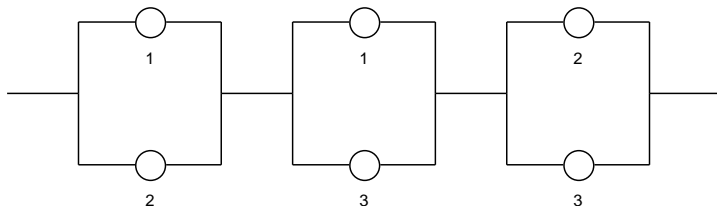


**Figure:** A reliability block diagram of a 2-out-of-3 system.

For a 2-out-of-3 system to function 2 out of 3 components must function. There are 3 possible subsets of components which contains 2 components:  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$ .



## A 2-out-of-3 system

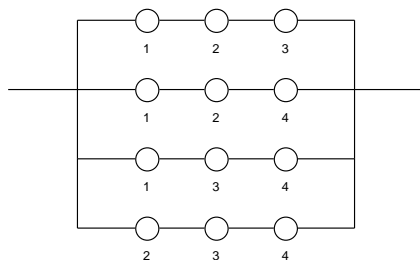


**Figure:** A reliability block diagram of a 2-out-of-3 system.

For a 2-out-of-3 system to fail 2 out of 3 components must fail. There are 3 possible subsets of components which contains 2 components:  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$ .



## A 3-out-of-4 system



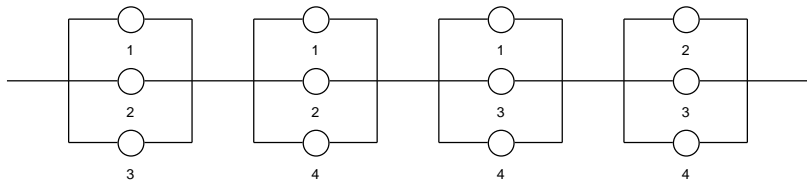
**Figure:** A reliability block diagram of a 3-out-of-4 system.

For a 3-out-of-4 system to function 3 out of 4 components must function. There are 4 possible subsets of components which contains 3 components:  $\{1, 2, 3\}$ ,  $\{1, 2, 4\}$ ,  $\{1, 3, 4\}$ ,  $\{2, 3, 4\}$ .





## A 2-out-of-4 system



**Figure:** A reliability block diagram of a 2-out-of-4 system.

For a 2-out-of-4 system to fail 3 out of 4 components must fail. There are 4 possible subsets of components which contains 3 components:  $\{1, 2, 3\}$ ,  $\{1, 2, 4\}$ ,  $\{1, 3, 4\}$ ,  $\{2, 3, 4\}$ .



# The reliability of a $k$ -out-of- $n$ system

In order to evaluate the reliability of a  $k$ -out-of- $n$  system it is convenient to introduce the following random variable:

$$S = \sum_{i=1}^n X_i.$$

Thus,  $S$  is the number of functioning components. This implies that:

$$h = P(\phi(\mathbf{X}) = 1) = P(S \geq k).$$



## The reliability of a $k$ -out-of- $n$ system (cont.)

If the component states are *independent*, and the component reliabilities are all *equal*, i.e.,  $p_1 = \dots = p_n = p$ , the random variable  $S$  is a binomially distributed random variable, and we have:

$$P(S = i) = \binom{n}{i} p^i (1 - p)^{n-i}.$$

Hence, the reliability of the system is given by:

$$h(\mathbf{p}) = h(p) = P(S \geq k) = \sum_{i=k}^n \binom{n}{i} p^i (1 - p)^{n-i}$$



## The reliability of a 2-out-of-3 system

EXAMPLE: Let  $(C, \phi)$  be a 2-out-of-3 system where the component states are independent, and where  $p_1 = p_2 = p_3 = p$ . We then have:

$$P(S = 2) = \binom{3}{2} p^2 (1 - p)^1 = 3p^2(1 - p)$$

$$P(S = 3) = \binom{3}{3} p^3 (1 - p)^0 = p^3.$$

Hence, the reliability of the system is:

$$h = P(S \geq 2) = 3p^2(1 - p) + p^3 = 3p^2 - 2p^3.$$



## The reliability of a 3-out-of-4 system

EXAMPLE: Let  $(C, \phi)$  be a 3-out-of-4 system where the component states are independent, and where  $p_1 = p_2 = p_3 = p_4 = p$ . We then have:

$$P(S = 3) = \binom{4}{3} p^3 (1 - p)^1 = 4p^3(1 - p)$$

$$P(S = 4) = \binom{4}{4} p^4 (1 - p)^0 = p^4.$$

Hence, the reliability of the system is:

$$h = P(S \geq 3) = 4p^3(1 - p) + p^4 = 4p^3 - 3p^4.$$



## The reliability of a $k$ -out-of- $n$ system (cont.)

When the component reliabilities are unequal, explicit analytical expressions for the distribution of  $S$  is not so easy to derive.

Let  $S$  be a stochastic variable with values in  $\{0, 1, \dots, n\}$ . We then define the *generating function* of  $S$  as:

$$G_S(y) = E[y^S] = \sum_{s=0}^n y^s P(S = s).$$

When a random variable  $S$  is the sum of a set of independent random variables  $X_1, \dots, X_n$ , the generating function of  $S$  is the product of the generating functions of  $X_1, \dots, X_n$ . By using this property it is possible to construct a very efficient algorithm for calculating the distribution of  $S$ .

We will return to this issue in an exercise.



## The reliability of a 2-out-of-3 system

EXAMPLE: Let  $(C, \phi)$  be a 2-out-of-3 system where the component states are independent with reliabilities  $p_1, p_2, p_3$ . We then have:

$$P(S = 2) = p_1 p_2 (1 - p_3) + p_1 (1 - p_2) p_3 + (1 - p_1) p_2 p_3$$

$$P(S = 3) = p_1 p_2 p_3$$

Hence, the reliability of the system is:

$$\begin{aligned} h &= p_1 p_2 (1 - p_3) + p_1 (1 - p_2) p_3 + (1 - p_1) p_2 p_3 + p_1 p_2 p_3 \\ &= p_1 p_2 + p_1 p_3 + p_2 p_3 - 2 p_1 p_2 p_3. \end{aligned}$$



## The reliability of a 3-out-of-4 system

EXAMPLE: Let  $(C, \phi)$  be a 3-out-of-4 system where the component states are independent with reliabilities  $p_1, p_2, p_3, p_4$ . We then have:

$$P(S = 3) = p_1 p_2 p_3 (1 - p_4) + p_1 p_2 (1 - p_3) p_4 \\ + p_1 (1 - p_2) p_3 p_4 + (1 - p_1) p_2 p_3 p_4$$

$$P(S = 4) = p_1 p_2 p_3 p_4$$

Hence, the reliability of the system is:

$$h = p_1 p_2 p_3 (1 - p_4) + p_1 p_2 (1 - p_3) p_4 \\ + p_1 (1 - p_2) p_3 p_4 + (1 - p_1) p_2 p_3 p_4 \\ + p_1 p_2 p_3 p_4 \\ = p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4 - 3 p_1 p_2 p_3 p_4$$





## Basic reliability calculation methods



## Section 3.1

# Pivotal decompositions



# Pivotal decompositions

## Theorem

*Let  $(C, \phi)$  be a binary monotone system. We then have:*

$$\phi(\mathbf{x}) = x_i \phi(1_i, \mathbf{x}) + (1 - x_i) \phi(0_i, \mathbf{x}), \quad i \in C. \quad (1)$$

*Similarly, for the reliability function of a binary monotone system where the component state variables are independent, we have*

$$h(\mathbf{p}) = p_i h(1_i, \mathbf{p}) + (1 - p_i) h(0_i, \mathbf{p}), \quad i \in C. \quad (2)$$



## Pivotal decompositions (cont.)

PROOF: Let  $i \in C$ , and consider two cases:

CASE 1.  $x_i = 1$ . Then the right-hand side of (1) becomes:

$$\phi(\mathbf{1}_i, \mathbf{x}).$$

Hence,  $\phi(\mathbf{x}) = \phi(\mathbf{1}_i, \mathbf{x})$ , so (1) holds in this case.

CASE 2.  $x_i = 0$ . Then the right-hand side of (1) becomes:

$$\phi(\mathbf{0}_i, \mathbf{x}),$$

Hence,  $\phi(\mathbf{x}) = \phi(\mathbf{0}_i, \mathbf{x})$ , so (1) holds in this case as well.

Equation (2) is proved by replacing the vector  $\mathbf{x}$  by  $\mathbf{X}$  in (1), and taking the expectation.



## Pivotal decompositions (cont.)

EXAMPLE: Let  $(C, \phi)$  be a 2-out-of-3 system where the component states are independent with reliabilities  $p_1, p_2, p_3$ . We then have:

$$\begin{aligned}\phi(\mathbf{x}) &= x_1\phi(1_1, \mathbf{x}) + (1 - x_1)\phi(0_1, \mathbf{x}) \\ &= x_1[x_2 \text{ II } x_3] + (1 - x_1)[x_2 \cdot x_3] \\ &= x_1[x_2 + x_3 - x_2 \cdot x_3] + (1 - x_1)[x_2 \cdot x_3]\end{aligned}$$

From this it follows that:

$$\begin{aligned}h(\mathbf{p}) &= p_1[p_2 + p_3 - p_2 \cdot p_3] + (1 - p_1)[p_2 \cdot p_3] \\ &= p_1p_2 + p_1p_3 - p_1p_2p_3 + p_2p_3 - p_1p_2p_3 \\ &= p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3\end{aligned}$$



## Pivotal decompositions (cont.)

### Corollary

*Let  $(C, \phi)$  be a binary monotone system. We then have:*

$$\begin{aligned}\phi(\mathbf{x}) &= x_i x_j \phi(1_i, 1_j, \mathbf{x}) + x_i (1 - x_j) \phi(1_i, 0_j, \mathbf{x}) \\ &\quad + (1 - x_i) x_j \phi(0_i, 1_j, \mathbf{x}) + (1 - x_i) (1 - x_j) \phi(0_i, 0_j, \mathbf{x}), \quad i, j \in C.\end{aligned}$$

*Similarly, for the reliability function of a binary monotone system where the component state variables are independent, we have:*

$$\begin{aligned}h(\mathbf{p}) &= p_i p_j h(1_i, 1_j, \mathbf{p}) + p_i (1 - p_j) h(1_i, 0_j, \mathbf{p}) \\ &\quad + (1 - p_i) p_j h(0_i, 1_j, \mathbf{p}) + (1 - p_i) (1 - p_j) h(0_i, 0_j, \mathbf{p}), \quad i, j \in C.\end{aligned}$$

PROOF: Use the pivotal decomposition theorem. Then apply the same theorem to  $\phi(1_i, \mathbf{x})$  and  $\phi(0_i, \mathbf{x})$ .



# Series and parallel components

## Definition

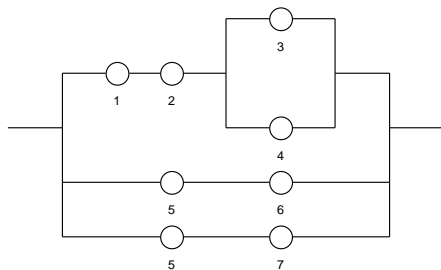
Let  $(C, \phi)$  be a binary monotone system, and let  $i, j \in C$ .

We say that  $i$  and  $j$  are *in series* if  $\phi$  depends on the component state variables,  $x_i$  and  $x_j$ , only through the product  $x_i \cdot x_j$ .

We say that  $i$  and  $j$  are *in parallel* if  $\phi$  depends on the component state variables,  $x_i$  and  $x_j$ , only through the coproduct  $x_i \amalg x_j$ .



## Series and parallel components (cont.)



In this system components 1 and 2 are in series, while components 3 and 4 are in parallel. Note, however, that components 5 and 6 are *not* in series since component 5 is also connected via component 7. Moreover, components 6 and 7 are in parallel.





## Series and parallel components (cont.)

### Theorem

*Let  $(C, \phi)$  be a binary monotone system, and let  $i, j \in C$ . Moreover, assume that the component state variables are independent.*

*If  $i$  and  $j$  are in series, then the reliability function,  $h$ , depends on  $p_i$  and  $p_j$  only through  $p_i \cdot p_j$ .*

*If  $i$  and  $j$  are in parallel, then the reliability function,  $h$ , depends on  $p_i$  and  $p_j$  only through  $p_i \amalg p_j$ .*



## Series and parallel components (cont.)

PROOF: If  $i$  and  $j$  are in series, we have:

$$\phi(1_i, 0_j, \mathbf{x}) = \phi(0_i, 1_j, \mathbf{x}) = \phi(0_i, 0_j, \mathbf{x}).$$

Thus, by pivotal decomposition we have:

$$\begin{aligned}\phi(\mathbf{x}) &= x_i x_j \phi(1_i, 1_j, \mathbf{x}) + x_i (1 - x_j) \phi(1_i, 0_j, \mathbf{x}) \\ &\quad + (1 - x_i) x_j \phi(0_i, 1_j, \mathbf{x}) + (1 - x_i) (1 - x_j) \phi(0_i, 0_j, \mathbf{x}) \\ &= (x_i x_j) \cdot \phi(1_i, 1_j, \mathbf{x}) + (1 - (x_i x_j)) \cdot \phi(0_i, 0_j, \mathbf{x}).\end{aligned}$$

Hence, by replacing the vector  $\mathbf{x}$  by  $\mathbf{X}$  and taking expectations we get:

$$h(\mathbf{p}) = (p_i p_j) \cdot h(1_i, 1_j, \mathbf{p}) + (1 - (p_i p_j)) \cdot h(0_i, 0_j, \mathbf{p}).$$

That is,  $h$ , depends on  $p_i$  and  $p_j$  only through  $p_i \cdot p_j$ .



## Series and parallel components (cont.)

If  $i$  and  $j$  are in parallel, we have:

$$\phi(1_i, 1_j, \mathbf{x}) = \phi(1_i, 0_j, \mathbf{x}) = \phi(0_i, 1_j, \mathbf{x}).$$

Thus, by pivotal decomposition we have:

$$\begin{aligned}\phi(\mathbf{x}) &= x_i x_j \phi(1_i, 1_j, \mathbf{x}) + x_i (1 - x_j) \phi(1_i, 0_j, \mathbf{x}) \\ &\quad + (1 - x_i) x_j \phi(0_i, 1_j, \mathbf{x}) + (1 - x_i) (1 - x_j) \phi(0_i, 0_j, \mathbf{x}) \\ &= (x_i \amalg x_j) \cdot \phi(1_i, 1_j, \mathbf{x}) + (1 - (x_i \amalg x_j)) \cdot \phi(0_i, 0_j, \mathbf{x}).\end{aligned}$$

Hence, by replacing the vector  $\mathbf{x}$  by  $\mathbf{X}$  and taking expectations we get:

$$h(\mathbf{p}) = (p_i \amalg p_j) \cdot h(1_i, 1_j, \mathbf{p}) + (1 - (p_i \amalg p_j)) \cdot h(0_i, 0_j, \mathbf{p}).$$

That is,  $h$ , depends on  $p_i$  and  $p_j$  only through  $p_i \amalg p_j$ .



## s-p-reductions

Consider a binary monotone system,  $(C, \phi)$  where the component state variables are independent, and let  $i, j \in C$ .

**SERIES REDUCTION:** If the components  $i$  and  $j$  are in series, then we may replace  $i$  and  $j$  by a single component  $i'$  with reliability  $p_{i'} = p_i p_j$  without altering the system reliability.

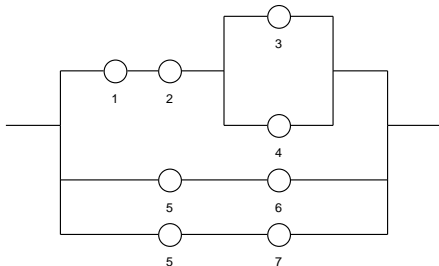
**PARALLEL REDUCTION:** If the components  $i$  and  $j$  are in parallel, then we may replace  $i$  and  $j$  by a single component  $i'$  with reliability  $p_{i'} = p_i \cup p_j$  without altering the system reliability.

Series and parallel reductions are referred to as *s-p-reductions*. Each s-p-reduction reduces the number of components in the system by one.

A system that can be reduced to a single component by applying a sequence of s-p-reductions is called an *s-p-system*.



## s-p-reductions (cont.)



This 7-component system is an s-p-system. Its reliability function can be derived using s-p-reductions *only* and is given by:

$$h(\mathbf{p}) = [p_1 p_2 (p_3 \text{ II } p_4)] \text{ II } [p_5 (p_6 \text{ II } p_7)]$$

