STK3405 – Week 35

A. B. Huseby & K. R. Dahl

Department of Mathematics
University of Oslo, Norway

k-out-of-n systems

A k-out-of-n system is a binary monotone system (C, ϕ) where $C = \{1, \ldots, n\}$ which functions if and only if at least k out of the n components are functioning.

Let the component state variable of component i be X_i, $i \in C$, and let the vector of component state variables be $\boldsymbol{X} = (X_1, \ldots, X_n)$.

The structure function, ϕ, can then be written:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i \geq k \\ 0 & \text{otherwise.} \end{cases}$$

An n-out-of-n system = A series system

An n-out-of-n system is the same as a series system:

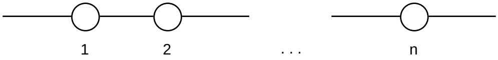


Figure: A reliability block diagram of an n-out-of-n system.

A 1-out-of-n system = A parallel system

A 1-out-of-n system is the same as a parallel system:

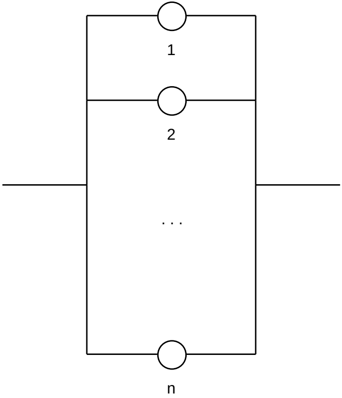


Figure: A reliability block diagram of an 1-out-of-n system.

A 3-out-of-4 system

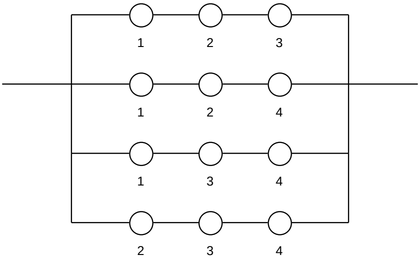


Figure: A reliability block diagram of a 3-out-of-4 system.

For a 3-out-of-4 system to function 3 out of 4 components must function. There are 4 possible subsets of components which contains 3 components: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$.

A 2-out-of-4 system

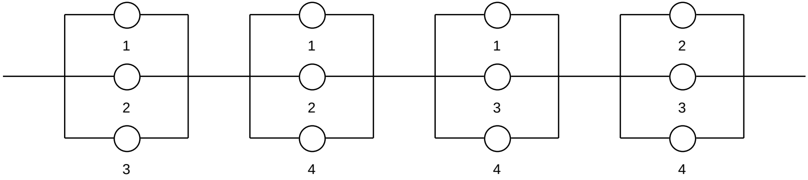


Figure: A reliability block diagram of a 2-out-of-4 system.

For a 2-out-of-4 system to fail 3 out of 4 components must fail. There are 4 possible subsets of components which contains 3 components: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$.

The reliability of a k-out-of-n system

In order to evaluate the reliability of a k-out-of-n system it is convenient to introduce the following random variable:

$$S = \sum_{i=1}^{n} X_i.$$

Thus, S is the number of functioning components. This implies that:

$$h = P(\phi(\boldsymbol{X}) = 1) = P(S \geq k).$$

The reliability of a k-out-of-n system (cont.)

If the component states are *independent*, and the component reliabilities are all *equal*, i.e., $p_1 = \cdots = p_n = p$, the random variable S is a binomially distributed random variable, and we have:

$$P(S = i) = \binom{n}{i} p^i (1-p)^{n-i}.$$

Hence, the reliability of the system is given by:

$$h(\boldsymbol{p}) = h(p) = P(S \geq k) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

The reliability of a 3-out-of-4 system

EXAMPLE: Let (C, ϕ) be a 3-out-of-4 system where the component states are independent, and where $p_1 = p_2 = p_3 = p_4 = p$. We then have:

$$P(S = 3) = \binom{4}{3} p^3 (1-p)^1 = 4p^3(1-p)$$

$$P(S = 4) = \binom{4}{4} p^4 (1-p)^0 = p^4.$$

Hence, the reliability of the system is:

$$h = P(S \geq 3) = 4p^3(1-p) + p^4 = 4p^3 - 3p^4.$$

Generating functions

Let S be a stochastic variable with values in $\{0, 1, \ldots, n\}$. We then define the *generating function* of S as:

$$G_S(y) = E[y^S] = \sum_{s=0}^{n} y^s P(S = s).$$

$G_S(y)$ is a polynomial because all the terms in the sum are of the form $a_s y^s$, $s = 0, 1, \ldots, n$.

The coefficient a_s is equal to $P(S = s)$.

Generating functions (cont.)

NOTE: A polynomial $g(y) = \sum_{s=0}^{n} a_s y^n$ is a generating function for a random variable S with values in $\{0, 1, \ldots n\}$ if and only if:

$$a_s \geq 0, \quad s = 0, 1, \ldots, n$$
$$\sum_{s=0}^{n} a_s = 1.$$

Generating functions (cont.)

Let T be another non-negative integer valued stochastic variable with values in $\{0, 1, \ldots, m\}$ which is independent of S. We then have that:

$$G_{S+T}(y) = G_S(y) \cdot G_T(y).$$

PROOF: By the definition of a generating function and the independence of S and T we have:

$$\begin{aligned} G_{S+T}(y) &= E[y^{S+T}] = E[y^S \cdot y^T] \\ &= E[y^S] \cdot E[y^T] \text{ (using that } S \text{ and } T \text{ are independent)} \\ &= G_S(y) \cdot G_T(y) \end{aligned}$$

Generating functions (cont.)

Let $X_1, \ldots, X_n$ be independent binary variables with $P(X_i = 1) = p_i$ and $P(X_i = 0) = 1 - p_i = q_i$, $i = 1, \ldots, n$. We then have that:

$$G_{X_i}(y) = q_i + p_i y, \quad i = 1, \ldots, n.$$

PROOF: By the definition of a generating function we get:

$$\begin{aligned} G_{X_i}(y) &= E[y^{X_i}] \\ &= y^0 \cdot P(X_i = 0) + y^1 P(X_i = 1) \\ &= q_i + p_i y, \qquad i = 1, \ldots, n. \end{aligned}$$

The reliability of a k-out-of-n system

Introduce:

$$S_j = \sum_{i=1}^{j} X_i, \quad j = 1, 2, \ldots, n,$$

and assume that we have computed $G_{S_j}(y)$. Thus, all the coefficients of $G_{S_j}(y)$ are known at this stage. We then compute:

$$G_{S_{j+1}}(y) = G_{S_j}(y) \cdot G_{X_{j+1}}(y).$$

The reliability of a k-out-of-n system (cont.)

Assume more specifically that:

$$G_{S_j}(y) = a_{j0} + a_{j1}y + a_{j2}y^2 + \cdots + a_{jj}y^j.$$

Then:

$$\begin{aligned} G_{S_{j+1}}(y) &= G_{S_j}(y) \cdot G_{X_{j+1}}(y) \\ &= (a_{j0} + a_{j1}y + a_{j2}y^2 + \cdots + a_{jj}y^j) \cdot (q_{i+1} + p_{i+1}y) \end{aligned}$$

In order to compute $G_{S_{j+1}}(y)$ we need to do $2(j+1)$ multiplications and j additions.

The reliability of a k-out-of-n system (cont.)

In order to compute $G_S(y) = G_{S_n}(y)$, $2 \cdot (2 + 3 + \cdots + n)$ multiplications and $(1 + 2 + \cdots (n-1))$ additions are needed. Thus, the number of operations grows roughly proportionally to n^2 operations.

Having calculated $G_S(y)$, a polynomial of degree n, the distribution of S is given by the coefficients of this polynomial.

If (C, ϕ) is a k-out-of-n-system with component state variables $X_1, \ldots, X_n$, then the reliability of this system is given by:

$$P(\phi(\boldsymbol{X}) = 1) = P(S \geq k) = \sum_{s=k}^{n} P(S = s)$$

Thus, the reliability of (C, ϕ) can be calculated in $O(n^2)$-time.

Pivotal decompositions

Theorem

Let (C, ϕ) be a binary monotone system. We then have:

$$\phi(\boldsymbol{x}) = x_i \phi(1_i, \boldsymbol{x}) + (1 - x_i)\phi(0_i, \boldsymbol{x}), \quad i \in C. \tag{1}$$

Similarly, for the reliability function of a binary monotone system where the component state variables are independent, we have

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p}), \quad i \in C. \tag{2}$$

Pivotal decompositions (cont.)

EXAMPLE: Let (C, ϕ) be a 2-out-of-3 system where the component states are independent with reliabilities p_1, p_2, p_3, We then have:

$$\begin{aligned}
\phi(\boldsymbol{x}) &= x_1\phi(1_1, \boldsymbol{x}) + (1 - x_1)\phi(0_1, \boldsymbol{x}) \\
&= x_1[x_2 \amalg x_3] + (1 - x_1)[x_2 \cdot x_3] \\
&= x_1[x_2 + x_3 - x_2 \cdot x_3] + (1 - x_1)[x_2 \cdot x_3]
\end{aligned}$$

From this it follows that:

$$\begin{aligned}
h(\boldsymbol{p}) &= p_1[p_2 + p_3 - p_2 \cdot p_3] + (1 - p_1)[p_2 \cdot p_3] \\
&= p_1p_2 + p_1p_3 - p_1p_2p_3 + p_2p_3 - p_1p_2p_3 \\
&= p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3
\end{aligned}$$

Pivotal decompositions (cont.)

Corollary

Let (C, ϕ) be a binary monotone system. We then have:

$$\phi(\boldsymbol{x}) = x_i x_j \phi(1_i, 1_j, \boldsymbol{x}) + x_i(1 - x_j)\phi(1_i, 0_j, \boldsymbol{x}) + (1 - x_i)x_j\phi(0_i, 1_j, \boldsymbol{x}) + (1 - x_i)(1 - x_j)\phi(0_i, 0_j, \boldsymbol{x}), \quad i, j \in C.$$

Similarly, for the reliability function of a binary monotone system where the component state variables are independent, we have:

$$h(\boldsymbol{p}) = p_i p_j h(1_i, 1_j, \boldsymbol{p}) + p_i(1 - p_j)h(1_i, 0_j, \boldsymbol{p}) + (1 - p_i)p_j h(0_i, 1_j, \boldsymbol{p}) + (1 - p_i)(1 - p_j)h(0_i, 0_j, \boldsymbol{p}), \quad i, j \in C.$$

Series and parallel components

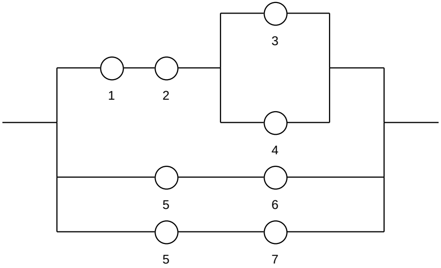
Definition

Let (C, ϕ) be a binary monotone system, and let $i, j \in C$.

We say that i and j are *in series* if ϕ depends on the component state variables, x_i and x_j, only through the product $x_i \cdot x_j$.

We say that i and j are *in parallel* if ϕ depends on the component state variables, x_i and x_j, only through the coproduct $x_i \amalg x_j$.

Series and parallel components (cont.)



In this system components 1 and 2 are in series, while components 3 and 4 are in parallel. Note, however, that components 5 and 6 are *not* in series since component 5 is also connected via component 7. Moreover, components 6 and 7 are in parallel.

Series and parallel components (cont.)

Theorem

Let (C, ϕ) be a binary monotone system, and let $i, j \in C$. Moreover, assume that the component state variables are independent.

If i and j are in series, then the reliability function, h, depends on p_i and p_j only through $p_i \cdot p_j$.

If i and j are in parallel, then the reliability function, h, depends on p_i and p_j only through $p_i \amalg p_j$.

s-p-reductions

Consider a binary monotone system, (C, ϕ) where the component state variables are independent, and let $i, j \in C$.

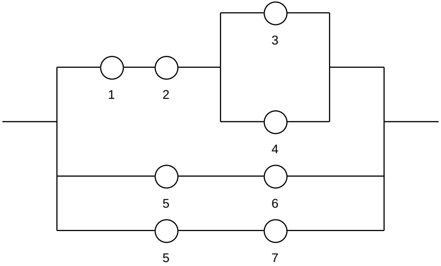
SERIES REDUCTION: If the components i and j are in series, then we may replace i and j by a single component i' with reliability $p_{i'} = p_i p_j$ without altering the system reliability.

PARALLEL REDUCTION: If the components i and j are in parallel, then we may replace i and j by a single component i' with reliability $p_{i'} = p_i \amalg p_j$ without altering the system reliability.

Series and parallel reductions are referred to as *s-p-reductions*. Each s-p-reduction reduces the number of components in the system by one.

A system that can be reduced to a single component by applying a sequence of s-p-reductions is called an *s-p-system*.

s-p-reductions (cont.)



This 7-component system is an s-p-system. Its reliability function can be derived using s-p-reductions *only* and is given by:

$$h(\boldsymbol{p}) = [p_1 p_2 (p_3 \amalg p_4)] \amalg [p_5 (p_6 \amalg p_7)]$$