STK3405 – Week 36a

A. B. Huseby & K. R. Dahl

Department of Mathematics
University of Oslo, Norway

Pivotal decompositions

Pivotal decompositions

Theorem

Let (C, ϕ) be a binary monotone system. We then have:

$$\phi(\boldsymbol{x}) = x_i \phi(1_i, \boldsymbol{x}) + (1 - x_i)\phi(0_i, \boldsymbol{x}), \quad i \in C. \tag{1}$$

Similarly, for the reliability function of a binary monotone system where the component state variables are independent, we have

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p}), \quad i \in C. \tag{2}$$

Series and parallel components

Definition

Let (C, ϕ) be a binary monotone system, and let $i, j \in C$.

We say that i and j are *in series* if ϕ depends on the component state variables, x_i and x_j, only through the product $x_i \cdot x_j$.

We say that i and j are *in parallel* if ϕ depends on the component state variables, x_i and x_j, only through the coproduct $x_i \amalg x_j$.

Series and parallel components (cont.)

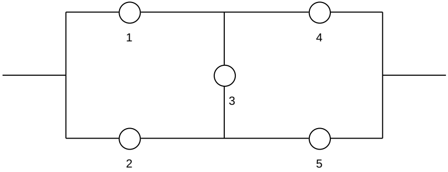
Theorem

Let (C, ϕ) be a binary monotone system, and let $i, j \in C$. Moreover, assume that the component state variables are independent.

If i and j are in series, then the reliability function, h, depends on p_i and p_j only through $p_i \cdot p_j$.

If i and j are in parallel, then the reliability function, h, depends on p_i and p_j only through $p_i \amalg p_j$.

Pivotal decompositions and s-p-reductions

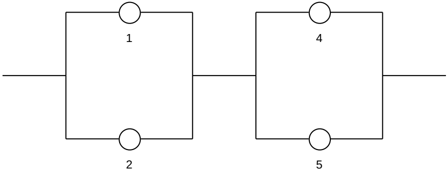


Let (C, ϕ) be the *bridge structure* shown above. In order to derive the structure function of this system, we note that:

$$\phi(1_3, \boldsymbol{X}) = \text{The system state given that component 3 is functioning}$$

$$\phi(0_3, \boldsymbol{X}) = \text{The system state given that component 3 is failed}$$

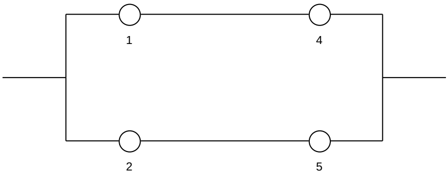
Pivotal decompositions and s-p-reductions (cont.)



Given that component 3 is functioning, the system becomes a series connection of two parallel systems. Hence, by using s-p-reductions, we get that:

$$\phi(1_3, \boldsymbol{X}) = (X_1 \amalg X_2) \cdot (X_4 \amalg X_5).$$

Pivotal decompositions and s-p-reductions (cont.)



Given that component 3 is failed, the system becomes a parallel connection of two series systems. Hence, by using s-p-reductions, we get that:

$$\phi(0_3, \boldsymbol{X}) = (X_1 \cdot X_4) \amalg (X_2 \cdot X_5).$$

Pivotal decompositions and s-p-reductions (cont.)

By the pivotal decomposition theorem it follows that ϕ can be written as:

$$\phi(\boldsymbol{X}) = X_3 \cdot \phi(1_3, \boldsymbol{X}) + (1 - X_3) \cdot \phi(0_3, \boldsymbol{X}).$$

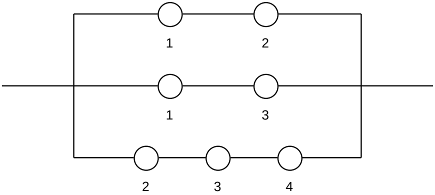
Combining all this we get that ϕ is given by:

$$\phi(\boldsymbol{X}) = X_3 \cdot (X_1 \amalg X_2)(X_4 \amalg X_5) + (1 - X_3) \cdot (X_1 \cdot X_4 \amalg X_2 \cdot X_5).$$

Moreover, assuming that the component state variables are independent, the reliability of the systems is:

$$h(\boldsymbol{p}) = p_3 \cdot (p_1 \amalg p_2)(p_4 \amalg p_5) + (1 - p_3) \cdot (p_1 \cdot p_4 \amalg p_2 \cdot p_5).$$

Pivotal decompositions and s-p-reductions

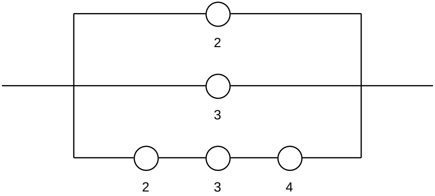


Let (C, ϕ) be the system shown above. In order to derive the structure function of this system, we note that:

$$\phi(1_1, \boldsymbol{X}) = \text{The system state given that component 1 is functioning}$$

$$\phi(0_1, \boldsymbol{X}) = \text{The system state given that component 1 is failed}$$

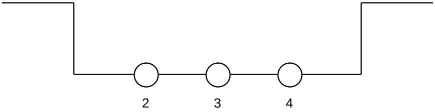
Pivotal decompositions and s-p-reductions (cont.)



Given that component 1 is functioning, the system becomes a parallel system of components 2 and 3 (since the lower path $\{2, 3, 4\}$ can be ignored in this case). Hence, by using s-p-reductions, we get that:

$$\phi(1_1, \boldsymbol{X}) = X_2 \amalg X_3.$$

Pivotal decompositions and s-p-reductions (cont.)



Given that component 1 is failed, the system becomes a series system of components 2, 3 and 4. Hence, by using s-p-reductions, we get that:

$$\phi(0_1, \boldsymbol{X}) = X_2 \cdot X_3 \cdot X_4.$$

Pivotal decompositions and s-p-reductions (cont.)

By the pivotal decomposition theorem it follows that ϕ can be written as:

$$\phi(\boldsymbol{X}) = X_1 \cdot \phi(1_1, \boldsymbol{X}) + (1 - X_1) \cdot \phi(0_1, \boldsymbol{X}).$$

Combining all this we get that ϕ is given by:

$$\phi(\boldsymbol{X}) = X_1 \cdot (X_2 \amalg X_3) + (1 - X_1) \cdot (X_2 \cdot X_3 \cdot X_4).$$

Moreover, assuming that the component state variables are independent, the reliability of the systems is:

$$h(\boldsymbol{p}) = p_1 \cdot (p_2 \amalg p_3) + (1 - p_1) \cdot (p_2 \cdot p_3 \cdot p_4).$$

Strict monotonicity

Theorem

Let $h(\boldsymbol{p})$ be the reliability function of a binary monotone system (C, ϕ) of order n, and assume that $0 < p_j < 1$ for all $j \in C$. If component i is relevant, then $h(\boldsymbol{p})$ is strictly increasing in p_i.

PROOF: Using pivotal decomposition wrt. component *i* it follows that:

$$\frac{\partial h(\boldsymbol{p})}{\partial p_i} = \frac{\partial}{\partial p_i}[p_i h(1_i, \boldsymbol{p}) + (1 - p_i)h(0_i, \boldsymbol{p})]$$

$$= h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p})$$

$$= E[\phi(1_i, \boldsymbol{X})] - E[\phi(0_i, \boldsymbol{X})] = E[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})]$$

$$= \sum_{(\cdot_i, \boldsymbol{X}) \in \{0,1\}^{n-1}} \big[\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})\big] P\big((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})\big)$$

Strict monotonicity (cont.)

Since ϕ is non-decreasing in each argument it follows that:

$$\left[\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})\right] \geq 0, \text{ for all } (\cdot_i, \boldsymbol{x}) \in \{0, 1\}^{n-1}.$$

If i is relevant, there exists at least one $(\cdot_i, \boldsymbol{y}) \in \{0, 1\}^{n-1}$ such that:

$$\left[\phi(1_i, \boldsymbol{y}) - \phi(0_i, \boldsymbol{y})\right] > 0.$$

Since $0 < p_j < 1$ for all $j \in C$, we have:

$$P\big((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})\big) > 0, \text{ for all } (\cdot_i, \boldsymbol{x}) \in \{0, 1\}^{n-1}.$$

From this it follows that:

$$\frac{\partial h(\boldsymbol{p})}{\partial p_i} > 0.$$

That is, $h(\boldsymbol{p})$ is strictly increasing in p_i.

Representation of binary monotone systems by paths and cuts

Path and cut sets

NOTATION: Let $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^n$. Then $\boldsymbol{y} < \boldsymbol{x}$ means that:

$$y_i \leq x_i, \text{ for all } i \in \{1, \ldots, n\}.$$

$$y_i < x_i, \text{ for at least one } i \in \{1, \ldots, n\}.$$

Let (C, ϕ) be a binary monotone system of order n. For a given vector $\boldsymbol{x} \in \{0,1\}^n$ the component set C can be divided into two subsets

$$C_0(\boldsymbol{x}) = \{i : x_i = 0\} = \text{The set of failed components}$$

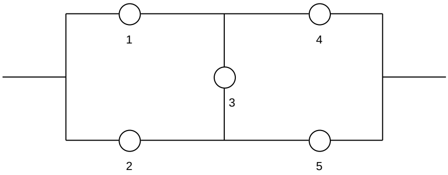
$$C_1(\boldsymbol{x}) = \{i : x_i = 1\} = \text{The set of functioning components}$$

Path and cut sets (cont.)

Let (C, ϕ) be a binary monotone system.

- A vector $\boldsymbol{x}$ is a *path vector* if and only if $\phi(\boldsymbol{x}) = 1$. The corresponding *path set* is $C_1(\boldsymbol{x})$.
- A *minimal path vector* is a path vector, $\boldsymbol{x}$, such that $\boldsymbol{y} < \boldsymbol{x}$ implies that $\phi(\boldsymbol{y}) = 0$. The corresponding *minimal path set* is $C_1(\boldsymbol{x})$.
- A vector $\boldsymbol{x}$ is a *cut vector* if and only if $\phi(\boldsymbol{x}) = 0$. The corresponding *cut set* is $C_0(\boldsymbol{x})$.
- A *minimal cut vector* is a cut vector, $\boldsymbol{x}$, such that $\boldsymbol{x} < \boldsymbol{y}$ implies that $\phi(\boldsymbol{y}) = 1$. The corresponding *minimal cut set* is $C_0(\boldsymbol{x})$.

Path and cut sets (cont.)



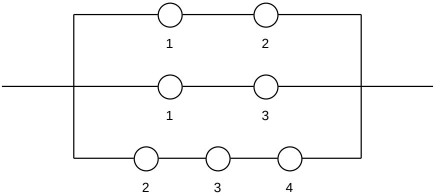
MINIMAL PATH SETS:

$$P_1 = \{1, 4\}, \quad P_2 = \{2, 5\}, \quad P_3 = \{1, 3, 5\}, \quad P_4 = \{2, 3, 4\}.$$

MINIMAL CUT SETS:

$$K_1 = \{1, 2\}, \quad K_2 = \{4, 5\}, \quad K_3 = \{1, 3, 5\}, \quad K_4 = \{2, 3, 4\}.$$

Path and cut sets (cont.)



MINIMAL PATH SETS:

$$P_1 = \{1, 2\}, \quad P_2 = \{1, 3\}, \quad P_3 = \{2, 3, 4\}.$$

MINIMAL CUT SETS:

$$K_1 = \{1, 2\}, \quad K_2 = \{1, 3\}, \quad K_3 = \{1, 4\}, \quad K_4 = \{2, 3\}.$$

Path and cut sets (cont.)

Consider a binary monotone system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$, and minimal cut sets $K_1, \ldots, K_k$.

For $j = 1, \ldots, p$ the j-th *minimal path series structure* is a binary monotone system (P_j, ρ_j) where:

$$\rho(\boldsymbol{x}^{P_j}) = \prod_{i \in P_j} x_i.$$

For $j = 1, \ldots, k$ the j-th *minimal cut parallel structure* is a binary monotone system (K_j, κ_j) where:

$$\kappa(\boldsymbol{x}^{K_j}) = \coprod_{i \in K_j} x_i.$$

Path and cut sets (cont.)

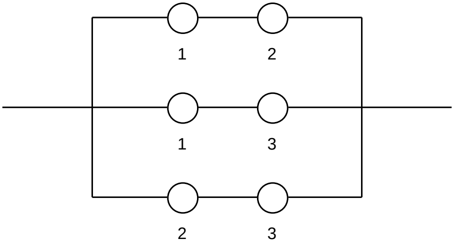
We now claim that:

$$\phi(\boldsymbol{x}) = \coprod_{j=1}^{p} \rho_j(\boldsymbol{x}^{P_j}) = \coprod_{j=1}^{p} \prod_{i \in P_j} x_i$$

$$= \prod_{j=1}^{k} \kappa_j(\boldsymbol{x}^{K_j}) = \prod_{j=1}^{k} \coprod_{i \in K_j} x_i$$

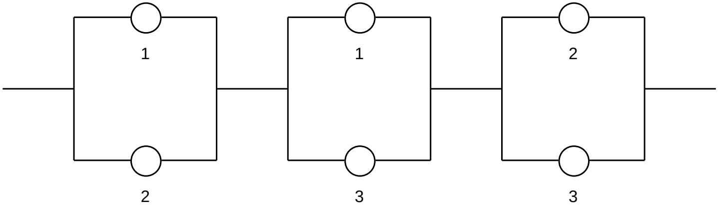
EXPLANATION: The system functions if and only if *at least one* of the minimal path series structures functions. Moreover, the system functions if and only if *all* the minimal cut series structures function.

Minimal path series structures of 2-out-of-3 system



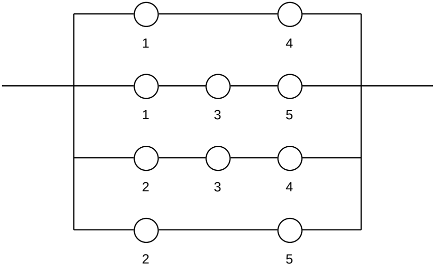
The minimal path sets of a 2-out-of-3 systems are : $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$, $P_3 = \{2, 3\}$.

Minimal cut parallel structures of 2-out-of-3 system



The minimal cut sets of a 2-out-of-3 systems are : $K_1 = \{1, 2\}$, $K_2 = \{1, 3\}$, $K_3 = \{2, 3\}$.

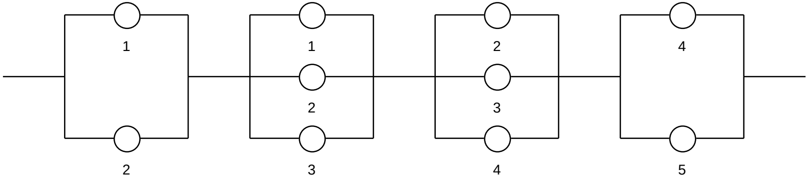
Minimal path series structures of a bridge system



The minimal path sets of a bridge systems are:

$$P_1 = \{1, 4\},\ P_2 = \{1, 3, 5\},\ P_3 = \{2, 3, 4\},\ P_4 = \{2, 5\}.$$

Minimal cut parallel structures of a bridge system



The minimal cut sets of a bridge systems are:

$$K_1 = \{1, 2\}, \; K_2 = \{1, 3, 5\}, \; K_3 = \{2, 3, 4\}, \; K_4 = \{4, 5\}.$$

Path and cut sets in dual systems

Theorem

Let (C, ϕ) be a binary monotone system, and let (C^D, ϕ^D) be its dual.

Then the following statements hold:

- $\boldsymbol{x}$ *is a path vector (alternatively, cut vector) for (C, ϕ) if and only if $\boldsymbol{x}^D$ is a cut vector (path vector) for (C^D, ϕ^D).*
- *A minimal path set (alternatively, cut set) for (C, ϕ) is a minimal cut set (path set) for (C^D, ϕ^D).*