STK3405 – Week 36b

A. B. Huseby & K. R. Dahl

Department of Mathematics
University of Oslo, Norway

Section 3.3

Modules of monotone systems

Modules of monotone systems

Let $A \subseteq C$. Then, the complement set of A, i.e., $C \setminus A$, is denoted by $\bar{A}$. We have the following formal definition of a module:

Definition

Let (C, ϕ) be a binary monotone system, and $A \subseteq C$. The monotone system (A, χ) is a *module* of (C, ϕ) if and only if the structure function ϕ can be written as:

$$\phi(\boldsymbol{x}) = \psi(\chi(\boldsymbol{x}^A), \boldsymbol{x}^{\bar{A}}), \qquad \text{for all } \boldsymbol{x} \in \{0, 1\}^n,$$

where ψ is a monotone structure function. The set A is called a *modular set* of (C, ϕ).

Modules of monotone systems

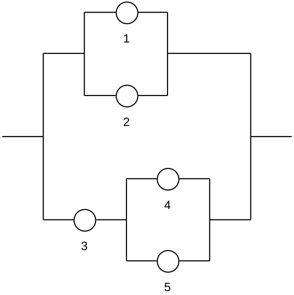
Definition

A *modular decomposition* of a monotone system (C, ϕ) is a set of modules $\{(A_j, \chi_j)\}_{j=1}^r$ connected by a binary monotone organisation structure function ψ. The following conditions must be satisfied:

- $C = \bigcup_{j=1}^r A_j$, and $A_j \cap A_k = \emptyset$ for $j \neq k$.
- $\phi(\boldsymbol{x}) = \psi[\chi_1(\boldsymbol{x}^{A_1}), \ldots, \chi_r(\boldsymbol{x}^{A_r})]$.

We observe that a modular decomposition is a disjoint partition of the component set into modules such that the structure function of the whole system is a function of the structure functions of these modules.

Modules of monotone systems (cont.)



Modules: (A_1, χ_1) and (A_2, χ_2) where $A_1 = \{1, 2\}$ and $A_2 = \{3, 4, 5\}$, and:

$$\chi_1(X_1, X_2) = X_1 \amalg X_2,$$

$$\chi_2(X_3, X_4, X_5) = x_3 \cdot (X_4 \amalg X_4)$$

$$\psi(\chi_1, \chi_2) = \chi_1 \amalg \chi_2$$

Section 3.4

Dynamic system analysis

Dynamic system analysis

Let (C, ϕ) be a binary monotone system, and introduce for $t \geq 0$:

$$\begin{aligned} X_i(t) &= \text{ the state of component } i \text{ at time } t,\ i \in C, \\ \phi(\boldsymbol{X}(t)) &= \text{ the state of the system at time } t. \end{aligned}$$

- $X_i(t)$ is a random variable (for any given t).
- $\{X_i(t)\}_{t \geq 0}$, is a stochastic process.
- $\phi(\boldsymbol{X}(t))$ is a random variable (for any given t).
- $\{\phi(\boldsymbol{X}(t))\}_{t \geq 0}$ is a stochastic process.

We assume that the stochastic processes $\{X_i(t), t \geq 0\}_{i=1}^n$ are independent.

Dynamic system analysis (cont.)

We also introduce:

$$p_i(t) = P(X_i(t) = 1) = \text{ The reliability of component } i \text{ at time } t,$$
$$h(\boldsymbol{p}(t)) = P(\phi(\boldsymbol{X}(t)) = 1) = \text{ The reliability of the system at time } t.$$

We assume that the components cannot be repaired and let:

$$T_i = \text{ The lifetime of component } i,$$
$$S = \text{ The lifetime of the system.}$$

NOTE:

$$P(X_i(t) = 1) = P(T_i > t), \quad i \in C,$$
$$P(\phi(\boldsymbol{X}(t)) = 1) = P(S > t).$$

Dynamic system analysis (cont.)

We denote the cumulative distribution of T_i by F_i, $i \in C$, and the cumulative distribution of ϕ by G. We then have the following relations:

$$p_i(t) = P(X_i(t) = 1) = P(T_i > t) = 1 - F_i(t) =: \bar{F}_i(t), \quad i \in C,$$

$$h(t) = P(\phi(\boldsymbol{X}(t)) = 1) = P(S > t) = 1 - G(t) =: \bar{G}(t).$$

NOTE: Determining the lifetime distribution for the system is the same as finding the reliability of the system at time t, i.e., $h(t)$, for all time $t \geq 0$, and then letting $G(t) = 1 - h(t)$.

Dynamic system analysis (cont.)

Theorem

For a monotone system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$ we have:

$$S = \begin{cases} \max_{1 \leq j \leq p} \min_{i \in P_j} T_i \\ \min_{1 \leq j \leq k} \max_{i \in K_j} T_i \end{cases}$$

PROOF: The lifetime of the system equals the lifetime of the minimal path series structure which lives the longest.

The lifetime of a minimal path series structure equals the lifetime of the shortest living component in this path set.

The second equality can be proved similarly.

Exact computation of reliability of binary monotone systems

Computational complexity

Let:

$$\begin{aligned} n &= \text{The size of the problem (e.g., number of components)} \\ t(n) &= \text{The worst case running time of the algorithm as a function of } n \\ f(n) &= \text{Some known non-negative increasing function of } n \end{aligned}$$

The order of the algorithm is said to be $O(f(n))$ if and only if there exists a positive constant M and a positive integer n_0 such that:

$$t(n) \leq Mf(n), \text{ for all } n \geq n_0.$$

If f is a polynomial in n, we say that the algorithm is a *polynomial time* algorithm, while if f is an exponential function of n, we say that the algorithm is an *exponential time* algorithm.

Computational complexity (cont.)

- NP (for nondeterministic polynomial time) is a complexity class used to describe certain types of problems.
- NP contains many important problems, the hardest of which are called *NP-complete* problems.
- Open question: Is it possible to find a polynomial time algorithm for solving NP-complete problems. Conjecture: *NO*.
- The class of *NP-hard* problems is a class of problems that are, informally, *at least as hard as the hardest problems in NP*.
- The problem of computing the reliability of a binary monotone system is known to be NP-hard in the general case.

Computational complexity (cont.)

EXAMPLE: In order to calculate the reliability of k-out-of-n system we need to do:

- $2 \cdot (2 + 3 + \cdots + n) = (n + 2)(n - 1)$ multiplications
- $1 + 2 + \cdots (n - 1) = \frac{n(n-1)}{2}$ additions

Thus, we have:

$$\begin{aligned} t(n) &= (n+2)(n-1) + \frac{n(n-1)}{2} \\ &= \frac{3}{2}n^2 + \frac{1}{2}n - 2 \leq 2n^2 \end{aligned}$$

This shows that the reliability of a k-out-of-n system can be calculated in $O(n^2)$ time.

Threshold systems

A *threshold system* is a binary monotone system (C, ϕ), where the structure function has the following form:

$$\phi(\boldsymbol{x}) = I(\sum_{i=1}^{n} a_i x_i \geq b),$$

where $a_1, \ldots, a_n$ and b are non-negative real numbers, and $I(\cdot)$ denotes the indicator function, i.e., a function defined for any event A which is 1 if A is true and zero otherwise.

NOTE: If $a_1 = \cdots = a_n = 1$ and $b = k$, the threshold system is reduced to a k-out-of-n system. Thus, threshold systems are a generalisation of k-out-of-n systems.

It can be shown that calculating the reliability of a threshold system in general is NP-hard.

Threshold systems (cont.)

Let (C, ϕ) a threshold system where $a_1, \ldots, a_n$ and b are positive integers, and introduce:

$$S_j = \sum_{i=1}^{j} a_i X_i, \quad j = 1, 2, \ldots, n.$$

By the assumptions it follows that $S_1, \ldots, S_n$ are integer valued stochastic variables.

Thus, the generating function for S_j, i.e., $G_{S_j}(y) = E[y^{S_j}]$ is a polynomial, and the distribution of S_j can be derived directly from the coefficients of $G_{S_j}(y)$, $j = 1, \ldots, n$.

Threshold systems (cont.)

We also introduce:

$$d_j = \sum_{i=1}^{j} a_i, \quad j = 1, 2, \ldots, n.$$

It follows that:

$$\deg(G_{S_j}(y)) = d_j, \quad j = 1, 2, \ldots, n.$$

Assuming $G_{S_j}(y)$ has been calculated, we can find $G_{S_{j+1}}(y)$ as:

$$G_{S_{j+1}}(y) = G_{S_j}(y) \cdot G_{a_{j+1} X_{j+1}}(y)$$

In the worst case this would require $2(d_j + 1)$ multiplications and d_j additions.

Threshold systems (cont.)

EXAMPLE: Assume that $a_j = 2^{j-1}$, $j = 1, \ldots, n$. We then have:

$$\deg(G_{S_j}(y)) = d_j = \sum_{i=1}^{j} 2^{i-1} = 2^j - 1, \quad j = 1, 2, \ldots, n.$$

In fact, in this case $G_{S_j}(y)$ consists of 2^j non-zero terms (including the constant term)!

Calculating $G_{S_{j+1}}(y)$ from $G_{S_j}(y)$ would require 2^{j+1} multiplications and $2^j - 1$ additions.

Thus, using generating functions for calculating the reliability of this threshold system takes $O(2^n)$ time.

Threshold systems (cont.)

EXAMPLE: Assume that $a_j \leq A$, $j = 1, \ldots, n$, where A is a fixed positive integer. We then have:

$$\deg(G_{S_j}(y)) = d_j \leq \sum_{i=1}^{j} A = Aj, \quad j = 1, 2, \ldots, n.$$

Calculating $G_{S_{j+1}}(y)$ from $G_{S_j}(y)$ would require at most $2(Aj + 1)$ multiplications and Aj additions.

Since A is a fixed constant, it follows that calculating the reliability of such a threshold system takes $O(n^2)$ time.