STK3405 – Week 37a

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Section 4.1.

State space enumeration

State space enumeration

We know that the reliability of a binary monotone system (C, ϕ) is simply the expected value of $\phi(\boldsymbol{X})$.

Thus, by standard probability theory this can be calculated as:

$$h = E[\phi(\boldsymbol{X})] = \sum_{\boldsymbol{x} \in \{0,1\}^n} \phi(\boldsymbol{x}) P(\boldsymbol{X} = \boldsymbol{x})$$

NOTE:

- Calculating reliability using state space enumeration implies summing 2^n terms.
- If the components are dependent, the entire joint distribution of $\boldsymbol{X}$ is needed.

State space enumeration (cont.)

If the component reliabilities are $p_1, \ldots, p_n$, we may write:

$$P(X_i = x_i) = p_i^{x_i}(1 - p_i)^{1-x_i}, \quad x_i = 0, 1, \quad i = 1, \ldots, n.$$

Thus, if the components are independent, we have:

$$P(\boldsymbol{X} = \boldsymbol{x}) = \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}$$

Inserting this into the state space enumeration formula we get:

$$h(\boldsymbol{p}) = \sum_{\boldsymbol{x} \in \{0,1\}^n} \phi(\boldsymbol{x}) \prod_{i=1}^{n} p_i^{x_i}(1 - p_i)^{1-x_i}.$$

State space enumeration (cont.)

NOTE: Due to the large number of terms in the expansion, the algorithm is at least of order $O(2^n)$. The order is typically greater since the work of computing $\phi(\cdot)$ for each $\boldsymbol{x}$ must be included as well.

It is possible to improve the algorithm by utilising that ϕ is non-decreasing in each argument:

If e.g., $\phi(\boldsymbol{x}_0) = 0$, it follows that $\phi(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} < \boldsymbol{x}_0$ as well. Hence, any state vector such that $\boldsymbol{x} < \boldsymbol{x}_0$ can be eliminated from the sum.

State space enumeration (cont.)

Let $C = \{1, 2, 3\}$, and let:

$$\phi(\boldsymbol{x}) = (x_1 \amalg x_2) \cdot x_3.$$

The component state variables are independent, and the component reliabilities are $P(X_i = 1) = p_i$, $i = 1, 2, 3$.

We then have:

$$\begin{aligned} h(\boldsymbol{p}) &= \phi(0,0,0) \cdot (1-p_1)(1-p_2)(1-p_3) \\ &\quad + \phi(1,0,0) \cdot p_1(1-p_2)(1-p_3) + \cdots + \phi(1,1,1) \cdot p_1 p_2 p_3 \\ &= p_1(1-p_2)p_3 + (1-p_1)p_2 p_3 + p_1 p_2 p_3. \end{aligned}$$

NOTE: There are exactly 3 vectors such that $\phi(\boldsymbol{x}) = 1$:

$$\boldsymbol{x}_1 = (1, 0, 1), \quad \boldsymbol{x}_2 = (0, 1, 1), \quad \boldsymbol{x}_3 = (1, 1, 1).$$

Hence, the reliability function contains just 3 terms, not $2^3 = 8$ terms.

NOTE: All terms have p_3 as a common factor, and:

$$p_1(1 - p_2) + (1 - p_1)p_2 + p_1 p_2 = p_1 \amalg p_2$$

Hence, $h(\boldsymbol{p})$ can be simplified to:

$$\begin{aligned} h(\boldsymbol{p}) &= [p_1(1 - p_2) + (1 - p_1)p_2 + p_1 p_2] \cdot p_3 \\ &= (p_1 \amalg p_2)p_3 \end{aligned}$$

The last expression can be obtained directly from the structure function since the system is an s-p-system.

The multiplication method

The multiplication method

Consider a binary monotone system with *minimal path sets*:

$$P_1, \ldots, P_p,$$

and *minimal cut sets*:

$$K_1, \ldots, K_k.$$

We then have:

$$\begin{aligned}\phi(\boldsymbol{X}) &= \coprod_{j=1}^{p} \prod_{i \in P_j} X_i = 1 - [\prod_{j=1}^{p} (1 - \prod_{i \in P_j} X_i)] \\ &= \prod_{j=1}^{k} \coprod_{i \in K_j} X_i = \prod_{j=1}^{k} [1 - \prod_{i \in K_j} (1 - X_i)].\end{aligned}$$

The multiplication method (cont.)

By expanding either the formula based on the minimal path sets, or the formula based on the minimal cut sets, and using that $X_i^r = X_i$, $i = 1, \ldots, n$, $r = 1, 2, \ldots$, we eventually get an expression of the form:

$$\phi(\boldsymbol{X}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} X_i$$

where for all $A \subseteq C$, $\delta(A)$ denotes the coefficient of the term associated with $\prod_{i \in A} X_i$. The δ-function is called the *signed domination function* of the structure.

The multiplication method (cont.)

By taking the expectation on both sides, and assuming that the component state variables are independent, we obtain:

$$h(\boldsymbol{p}) = E[\phi(\boldsymbol{X})] = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} E[X_i] = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} p_i$$

The sum contains 2^n terms.

Hence, the multiplication method also has at least order $O(2^n)$.

Typically, many of the terms vanish since we may have $\delta(A) = 0$ for many of the sets A. Improved versions of this method utilizes this.

The multiplication method (cont.)

Introduce the following version of the structure function of a binary monotone system (C, ϕ):

$$\phi(B) = \phi(\mathbf{1}^B, \mathbf{0}^{\bar{B}}), \text{ for all } B \subseteq C.$$

Thus, $\phi(B)$ represents the state of the system given that all the components in the set B are functioning, while all the components in the set $\bar{B} = C \setminus B$ are failed.

We now have:

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B), \text{ for all } A \subseteq C.$$

NOTE: This formula allows us to compute the signed domination function without using the minimal path and cut sets. For certain classes of systems this can be used to establsih formulas allowing much faster calculations.

Properties of the signed domination

Theorem

Let (C, ϕ) be a binary monotone system, and let $A \subseteq C$.

If $\phi(A) = 0$, then $\delta(A) = 0$ as well.

PROOF: Since ϕ is nondecreasing in each argument, it follows from the assumption that $\phi(B) = 0$ for all $B \subseteq A$. Hence,

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B) = 0.$$

Properties of the signed domination (cont.)

Theorem

Let (C, ϕ) be a binary monotone system, let $A \subseteq C$, and let $i \in A$. If $\phi(B \cup i) = \phi(B)$ for all $B \subseteq A \setminus i$, then $\delta(A) = 0$.

PROOF:

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B)$$

$$= \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B|} \phi(B) + \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B \cup i|} \phi(B \cup i)$$

$$= \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B|} \phi(B) - \sum_{B \subseteq A \setminus i} (-1)^{|A|-|B|} \phi(B) = 0$$

Properties of the signed domination (cont.)

Theorem

Let (C, ϕ) be a binary monotone system, let $P \subseteq C$ be a minimal path set.

Then $\delta(P) = 1$.

PROOF: Since P is a minimal path set, it follows that $\phi(P) = 1$ and that $\phi(B) = 0$ for all $B \subset P$. Hence,

$$\delta(P) = \sum_{B \subseteq P} (-1)^{|P|-|B|} \phi(B)$$

$$= (-1)^{|P|-|P|} \phi(P) = 1.$$