STK3405 – Chapter 6, part 1

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Association and bounds for the system reliability

Section 6.1

Associated random variables

Associated random variables

Definition (Associated random variables)

Let $T_1, \ldots, T_n$ be random variables, and let $\boldsymbol{T} = (T_1, \ldots, T_n)$. We say that $T_1, \ldots, T_n$ are associated if

$$\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0,$$

for all binary non-decreasing functions Γ and Δ.

NOTE: We only require $\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0$ for all *binary* non-decreasing functions.

Associated random variables (cont.)

Theorem (Generalized covariance property)

Let $T_1, \ldots, T_n$ be associated random variables, and f and g functions which are non-decreasing in each argument such that $\mathrm{Cov}(f(\boldsymbol{T}), g(\boldsymbol{T}))$ exists, i.e.,

$$E[|f(\boldsymbol{T})|] < \infty, E[|g(\boldsymbol{T})|] < \infty, E[|f(\boldsymbol{T})g(\boldsymbol{T})|] < \infty.$$

Then we have:

$$\mathrm{Cov}(f(\boldsymbol{T}), g(\boldsymbol{T})) \geq 0.$$

Associated random variables (cont.)

Theorem (Properties of Associated variables)

Associated random variables have the following properties:

(*i*) *Any subset of a set of associated random variables also consists of associated random variables.*

(*ii*) *A single random variable is always associated.*

(*iii*) *Non-decreasing functions of associated random variables are associated.*

(*iv*) *If two sets of associated random variables are independent, then their union is a set of associated random variables.*

Associated random variables (cont.)

Theorem (Independence)

Let $T_1, \ldots, T_n$ be independent. Then, they are also associated.

PROOF: (Induction on n.) The result obviously holds for $n = 1$ by property (*ii*).

Assume that the theorem holds for $n = m - 1$. That is, $\{T_1, \ldots, T_{m-1}\}$ is a set of associated random variables.

Moreover, by property (*ii*), $\{T_m\}$ is associated as well.

By the assumption, these two sets are independent. Hence, it follows from property (*iv*) that their union $\{T_1, \ldots, T_{m-1}, T_m\}$ is a set of associated random variables.

Thus, the result is proved by induction.

Associated random variables (cont.)

Theorem (Absolute dependence)

Let $T_1, \ldots, T_n$ be completely positively dependent random variables, i.e.,

$$P(T_1 = T_2 = \ldots = T_n) = 1.$$

Then they are associated.

PROOF: Let Γ, Δ be binary functions which are non-decreasing in each argument and let $\boldsymbol{T} = (T_1, \ldots, T_n)$ and $\boldsymbol{T}_1 = (T_1, \ldots, T_1)$. By the assumption it follows that $\boldsymbol{T}$ and $\boldsymbol{T}_1$ must have the same distribution. Hence, we get that:

$$\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) = \mathrm{Cov}(\Gamma(\boldsymbol{T}_1), \Delta(\boldsymbol{T}_1)) \geq 0$$

where the final inequality follows by property (ii) and the definition.

Associated random variables (cont.)

Theorem (Association of paths and cuts)

Let $X_1, \ldots, X_n$ be the associated or independent component state variables of a monotone system (C, ϕ). Moreover, let the minimal path series structures of the system be $(P_1, \rho_1), \ldots, (P_p, \rho_p)$, where:

$$\rho_j(\boldsymbol{X}^{P_j}) = \prod_{i \in P_j} X_i, \quad j = 1, \ldots, p.$$

Then, $\rho_1, \ldots, \rho_p$ are associated.

Similarly, let the minimal cut parallel structures of the system be $(K_1, \kappa_1), \ldots, (K_k, \kappa_k)$, where:

$$\kappa_j(\boldsymbol{X}^{K_j}) = \coprod_{i \in K_j} X_i, \quad j = 1, \ldots, k.$$

Then, $\kappa_1, \ldots, \kappa_k$ are associated.

Associated random variables (cont.)

Theorem (Extension of property (iii))

Let $\boldsymbol{T} = (T_1, \ldots, T_n)$ be associated, and let:

$$U_i = g_i(\boldsymbol{T}), \quad i = 1, \ldots, m,$$

where g_i, $i = 1, \ldots, m$ are non-increasing functions. Then, $\boldsymbol{U} = (U_1, \ldots, U_m)$ is associated.

Associated random variables (cont.)

PROOF: Let Γ, Δ be binary non-decreasing functions, and introduce $\boldsymbol{U} = \boldsymbol{g}(\boldsymbol{T}) = (g_1(\boldsymbol{T}), \ldots, g_m(\boldsymbol{T}))$. Then let:

$$\bar{\Gamma}(\boldsymbol{T}) = 1 - \Gamma(\boldsymbol{g}(\boldsymbol{T})) = 1 - \Gamma(\boldsymbol{U})$$

$$\bar{\Delta}(\boldsymbol{T}) = 1 - \Delta(\boldsymbol{g}(\boldsymbol{T})) = 1 - \Delta(\boldsymbol{U})$$

It follows that $\bar{\Gamma}$ and $\bar{\Delta}$ are binary and non-decreasing in T_i, $i = 1, \ldots, n$. Since $\boldsymbol{T}$ is associated, it follows that:

$$\begin{aligned}
\mathrm{Cov}(\Gamma(\boldsymbol{U}), \Delta(\boldsymbol{U}))) &= \mathrm{Cov}(1 - \bar{\Gamma}(\boldsymbol{T}), 1 - \bar{\Delta}(\boldsymbol{T})) \\
&= \mathrm{Cov}(1, 1) + \mathrm{Cov}(1, -\bar{\Delta}(\boldsymbol{T})) + \mathrm{Cov}(-\bar{\Gamma}(\boldsymbol{T}), 1) \\
&+ \mathrm{Cov}(-\bar{\Gamma}(\boldsymbol{T}), -\bar{\Delta}(\boldsymbol{T})) \\
&= \mathrm{Cov}(\bar{\Gamma}(\boldsymbol{T}), \bar{\Delta}(\boldsymbol{T})) \geq 0.
\end{aligned}$$

Hence, we conclude that $\boldsymbol{U}$ is associated.

Associated random variables (cont.)

Theorem (Bivariate association)

Let X and Y be two binary random variables. Then, X and Y are associated if and only if

$$\mathrm{Cov}(X, Y) \geq 0.$$

PROOF: Assume first that X and Y are associated. We may then choose $\Gamma(X, Y) = X$ and $\Delta(X, Y) = Y$.

Since obviously Γ and Δ are binary and non-decreasing functions, it follows from def: associated that $\mathrm{Cov}(X, Y) = \mathrm{Cov}(\Gamma, \Delta) \geq 0$.

Associated random variables (cont.)

Assume conversely that $\text{Cov}(X, Y) \geq 0$. We want to prove that this implies that $\text{Cov}(\Gamma(X, Y), \Delta(X, Y)) \geq 0$ for all binary non-decreasing functions, Γ and Δ.

The only choices for Γ and Δ are :

$$\Gamma_1 \equiv 0, \quad \Gamma_2 = X \cdot Y, \quad \Gamma_3 = X, \quad \Gamma_4 = Y, \quad \Gamma_5 = X \amalg Y, \quad \Gamma_6 \equiv 1.$$

These functions can be ordered as follows:

$$\Gamma_1 \leq \Gamma_2 \leq \left\{ \begin{array}{c} \Gamma_3 \\ \Gamma_4 \end{array} \right\} \leq \Gamma_5 \leq \Gamma_6.$$

Associated random variables (cont.)

Assume first that Γ and Δ from the set $\{\Gamma_1, \ldots, \Gamma_6\}$ such that $\Gamma(X, Y) \leq \Delta(X, Y)$. We then have:

$$\begin{aligned}\mathrm{Cov}(\Gamma, \Delta) &= E(\Gamma \cdot \Delta) - E(\Gamma) \cdot E(\Delta) \\ &= E(\Gamma) - E(\Gamma) \cdot E(\Delta) = E(\Gamma)[1 - E(\Delta)] \geq 0.\end{aligned}$$

The only possibility left is $\Gamma = \Gamma_3 = X$ and $\Delta = \Gamma_4 = Y$. However, in this case we get that:

$$\mathrm{Cov}(\Gamma, \Delta) = \mathrm{Cov}(X, Y) \geq 0,$$

where the last inequality follows by the assumption. Hence, we conclude that $\mathrm{Cov}(\Gamma, \Delta) \geq 0$ for all binary non-decreasing functions, and thus the result is proved.