14th September, 2020

STK3405

Mandatory assignment 1 of 2

Submission deadline

Thursday 1st October 2020, 14:30 at Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

About the mandatory assignment for STK3405

In order to get your assignment approved, you must have at least 65 % correct answers *on both problems.*

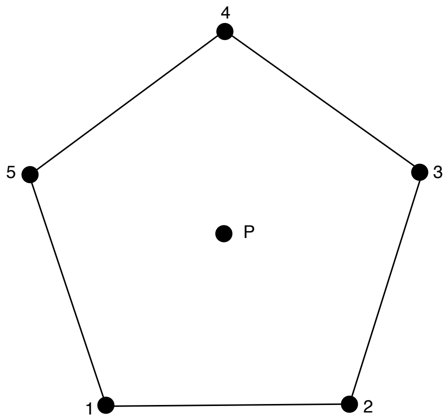

Figure 1: Et system for styring av en elektronstråle.

Problem 1. In this test we consider a steering system used to control an electron beam (similar to systems used in TV sets). The system consists of five electromagnets located at the corners of an equilateral pentagon. The electron beam originates from a source behind the pentagon and passes through the pentagon near its center. A very simplified sketch of the system is shown in Figure 1.

In the figure, the components of the system, i.e., the electromagnets, are denoted bywhile the point where the beam passes through is denoted by P. It is not necessary that all the electromagnets are functioning to ensure the ability to steer the electron beam. A necessary and sufficient condition is that the point P is contained within the polygon spanned by the corners of the pentagon corresponding to the functioning components. This implies that at least three components are needed to make the system work. Still, not all possible combinations of three components are sufficient. Observe e.g., that the triangle spanned by the corners 1, 2 and 3, do not contain P. On the other hand if components 1, 2 and 4 are working, then this is sufficient since the triangle spanned by these three corners indeed contains P.

(a) Find the minimal path sets of the system (5 sets), and the minimal cut sets of the system (5 sets).

(b) Show that the structure function of the system can be written as:

$$\begin{aligned}\phi(\mathbf{X}) = {} & X_1X_2X_4 + X_2X_3X_5 + X_1X_3X_4 + X_2X_4X_5 + X_1X_3X_5 \\ & - X_1X_2X_3X_4 - X_1X_2X_3X_5 - X_1X_2X_4X_5 - X_1X_3X_4X_5 - X_2X_3X_4X_5 \\ & + X_1X_2X_3X_4X_5,\end{aligned}$$

where $X_1, \ldots, X_5$ are the component state variables.

(c) Observe that all the coefficients in the expression for ϕ are either $+1$ or -1. Describe another class of systems with this property.

(d) Assume at this stage that all the components are stochastically independent, and that all the components have reliability p. Show that the system reliability, h, can be written as:

$$h(p) = 5p^3 - 5p^4 + p^5$$

(e) Assume in particular that $p = 0.75$. Compute h for this value of p. [Answer: 0.7646]

(f) Make a plot of how h varies as a function of p. For which values of p do we have $h > p$? [No exact calculations are required here. It is sufficient if one uses the plot to obtain an approximate solution to this question.]

In the remaining part of this test we assume that the electromagnets can fail in two ways: either the magnets burn and stop working as a result of this, or the magnets fail as a result of an electric power failure. If the electric power fails, then all the components fail. We then introduce for $i = 1, \dots, 5$:

$$Y_i = \begin{cases} 1 & \text{if the } i\text{th magnet has not burned} \\ 0 & \text{otherwise} \end{cases}$$

og:

$$Z = \begin{cases} 1 & \text{if the electrical power has not failed} \\ 0 & \text{otherwise} \end{cases}$$

We assume that $Y_1, \dots Y_5$ and Z are stochastically independent and that $P(Y_i = 1) = \alpha$, $i = 1, \dots, 5$ while $P(Z = 1) = \theta$.

(g) Explain briefly why this implies that:

$$P(X_i = 1) = \alpha \cdot \theta, \quad i = 1, \dots, 5.$$

(h) Compute $\mathrm{Cov}(X_i, X_j)$ for $i \neq j$.

(i) Finally, compute the system reliability expressed as a function of α and θ.

Problem 2. In this problem we consider a subsea pipeline system for transporting oil. The probability that there are no holes in a given pipeline is given by:

$$P(\ell) = \exp(-\lambda \ell), \tag{1}$$

where $\lambda > 0$ is a fixed given number, and $\ell > 0$ denotes the length of the pipeline.

(a) Let $\Delta\ell$ be a positive number close to 0. Using Taylor series expansion of the function P above it follows that:

$$1 - P(\Delta\ell) \approx \lambda \Delta\ell.$$

Use this to provide a practical interpretation of the parameter λ.

(b) Assume that we have a pipeline of the above type of length L. We then imagine that this pipeline is partitioned into n disjoint parts such that:

$$\ell_i = \text{ the length of the } i\text{th part}, \quad i = 1, \dots, n.$$

The formula (1) is assumed to be valid for each of the n parts, while the total length of the pipeline is:

$$L = \sum_{i=1}^{n} \ell_i.$$

We then introduce for $i = 1, \dots n$:

$$X_i = \begin{cases} 1 & \text{if there are no holes in the } i\text{th part} \\ 0 & \text{otherwise} \end{cases}$$

og:

$$\phi = \begin{cases} 1 & \text{if there are no holes in the pipeline} \\ 0 & \text{otherwise} \end{cases}$$

Explain briefly why:

$$\phi = \phi(X_1, \dots, X_n) = \prod_{i=1}^{n} X_i,$$

and:

$$P(\phi = 1) = \prod_{i=1}^{n} P(X_i = 1).$$

(c) In order to transport the oil from a source S to a terminal T two pipelines of the same type as above are used. The pipelines both have length L, and run in parallel between S and T. In case there is a hole in one pipeline, this will be closed on both sides of the hole to prevent leakage.

We assume that both pipelines are functioning independently of each other. Do you think this is a reasonable assumption? Explain why you think so.

We consider the two pipelines as a coherent system, and say that the system is functioning if there is at least one intact pipeline between S and T. The components in this system are the two pipelines. Calculate the reliability of the system.

(d) In order to increase the reliability of the system, one connects a bridge pipeline at the point U, located somewhere between S and T. We denote the distance beween S and U by y. Thus, the distance between U and T then becomes $(L - y)$. The probability that the bridge is functioning is assumed to be q. Draw a figure ilustrating the structure of this new system. In particular, list the components of this system, and write down the reliability of each of the components.

We assume that the bridge and the pipeline parts function independently of each other, and that the bridge does not change the reliability of the other pipelines.

Show that the reliability of this system expressed in terms of y and q are given by:

$$h_1(y, q) = qe^{-\lambda L}(2 - e^{-\lambda y})(2 - e^{-\lambda(L-y)}) + (1 - q)e^{-\lambda L}(2 - e^{-\lambda L}).$$

(e) Determine the value of y that maximizes the system reliability. Calculate $h_1(y, q)$ for this particular vale of y.

(f) Assume that we, instead of a bridge at U, are given the option of improving the quality of the pipelines. More specifically, we assume that if higher quality pipelines are used, this implies that the parameter λ is replaced by $\lambda^* = \lambda/2$. Show that the reliability of the system in this case becomes:

$$h_2 = e^{-\lambda^* L}(2 - e^{-\lambda^* L}) = e^{-\lambda L/2}(2 - e^{-\lambda L/2}).$$

(g) Finally, assume that $q \approx 1$, and that the bridge is placed so that y ets the optimal value found in (e). Compare h_1 og h_2. Which alternative would you recommend if the price of the bridge is the same as the price of improving the pipeline quality?