UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK3405/4405 — Introduction to risk and reliability analysis
Day of examination:	Friday December 17th 2021.
Examination hours:	09.00 – 13.00.
This problem set consists of 4 pages.	
Appendices:	None.
Permitted aids:	Approved calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All subpoints will be equally weighted in the marking.

Problem 1

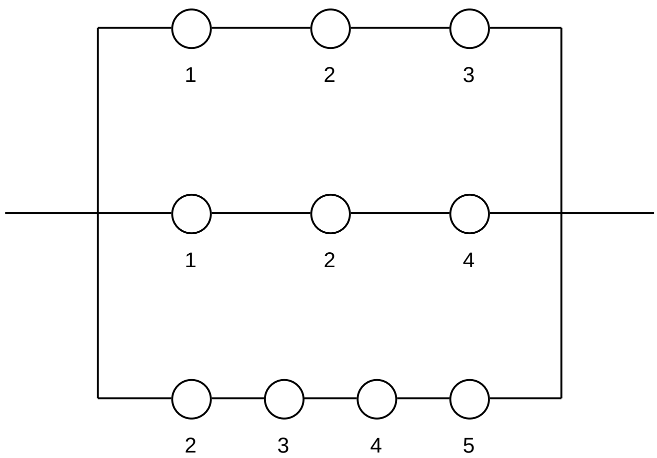

Figure 1: Block diagram of (C, ϕ)

Consider the binary monotone system (C, ϕ) shown in Figure 1. The component set of the system is $C = \{1, 2, \ldots, 5\}$. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_5)$ denote the vector of component state variables, and assume throughout this problem that $X_1, X_2, \ldots, X_5$ are stochastically independent. We also let $\boldsymbol{p} = (p_1, p_2, \ldots, p_5)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, \ldots, 5$.

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a) Find the minimal path sets (3 sets) and the minimal cut sets (5 sets) of the system.

b) We let $h(\boldsymbol{p}) = P(\phi = 1)$ denote the reliability function of the system. Show that:

$$h(\boldsymbol{p}) = p_2 \cdot [p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5]$$

The Birnbaum measure for the *reliability importance* of component i is defined as:

$$I_B^{(i)} = P(\text{Component } i \text{ is critical for the system}), \quad i = 1, 2, \ldots, 5.$$

c) Show that:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \quad i = 1, 2, \ldots, 5.$$

d) Show that:

$$I_B^{(2)} = p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5$$
$$I_B^{(5)} = (1 - p_1) \cdot p_2 p_3 p_4$$

In the remaining part of this problem we assume that $0 < p_i < 1$, $i = 1, 2, \ldots, 5$.

e) Show that if $p_5 \geq p_2$, then $I_B^{(2)} > I_B^{(5)}$.

f) Show that if $p_1 \geq \frac{1}{2}$, then $I_B^{(2)} > I_B^{(5)}$.

g) In this point we assume more specifically that $p_1 = p_5 = \frac{1}{10}$ and that $p_2 = p_3 = p_4 = \frac{9}{10}$. Calculate $I_B^{(2)}$ and $I_B^{(5)}$ and compare the results. Comment your findings.

The Birnbaum measure for the *structural importance* of component i is defined as:

$$J_B^{(i)} = \frac{1}{2^{5-1}} \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})], \quad i = 1, 2, \ldots, 5.$$

h) Explain briefly why: $J_B^{(2)} > J_B^{(i)}$ for $i = 1, 3, 4, 5$.

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Problem 2

Let $X_1, \ldots, X_n$ be n binary associated random variables.

a) Show that:

$$E[\prod_{i=1}^{n} X_i] \;\geq\; \prod_{i=1}^{n} E[X_i] \tag{1}$$

$$E[\coprod_{i=1}^{n} X_i] \;\leq\; \coprod_{i=1}^{n} E[X_i] \tag{2}$$

Let $X_1, \ldots, X_n$ be the associated component state variables of a binary monotone system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$.

b) Show that:

$$\max_{1 \leq j \leq p} \prod_{i \in P_j} E[X_i] \;\leq\; E[\phi] \;\leq\; \min_{1 \leq j \leq k} \coprod_{i \in K_j} E[X_i] \tag{3}$$

c) Show that:

$$\prod_{j=1}^{k} E[\coprod_{i \in K_j} X_i] \;\leq\; E[\phi] \;\leq\; \coprod_{j=1}^{p} E[\prod_{i \in P_j} X_i]. \tag{4}$$

We denote the lower and upper bounds on $E[\phi]$ given in (3) by L_1 and U_1 respectively. Similarly, we denote the lower and upper bounds on $E[\phi]$ given in (4) by L_2 and U_2 respectively.

In the rest of this problem we assume that (C, ϕ) is a 2-out-of-3 system. That is, $C = \{1, 2, 3\}$ and the structure function, ϕ, is given by:

$$\phi(\boldsymbol{X}) = X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3,$$

where $\boldsymbol{X} = (X_1, X_2, X_3)$. Moreover, we assume that the joint distribution of the component state variables satisfies the following properties:

$$E[X_1] = E[X_2] = E[X_3] = p,$$

$$E[X_1X_2] = E[X_1X_3] = E[X_2X_3] = p^{2-\alpha},$$

$$E[X_1X_2X_3] = p^{3-2\alpha},$$

where $0 < p < 1$ and $0 \leq \alpha \leq 1$. It can be shown that these properties imply that X_1, X_2, X_3 are *associated random variables.*

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d) We now consider the correlation between the component state variables. Show that:

$$\text{Corr}(X_i, X_j) = \frac{p^{2-\alpha} - p^2}{p(1-p)}, \qquad \text{for } i \neq j.$$

Moreover, show that the correlation is increasing in α. In particular, calculate the correlation for the cases $\alpha = 0$ and $\alpha = 1$. Comment your findings.

e) Show that:

$$L_1 = p^2 \quad \text{and} \quad U_1 = 1 - (1-p)^2$$

and that:

$$L_2 = (2p - p^{2-\alpha})^3 \quad \text{and} \quad U_2 = 1 - (1 - p^{2-\alpha})^3$$

and that:

$$E[\phi] = 3p^{2-\alpha} - 2p^{3-2\alpha}$$

f) Show that L_2 is decreasing in α while U_2 is increasing in α. What can you say about the quality of these bounds when the correlation between the component state variables increases?

g) Assume that $\alpha = 1$. Show that we in this case have:

$$L_2 \;<\; L_1 \;<\; E[\phi] \;<\; U_1 \;<\; U_2$$

Which bounds would you recommend in this case?

h) Assume that $\alpha = 0$. What kind of bounds would you recommend in this case?

END