Exam STK3405/4405 - 2015

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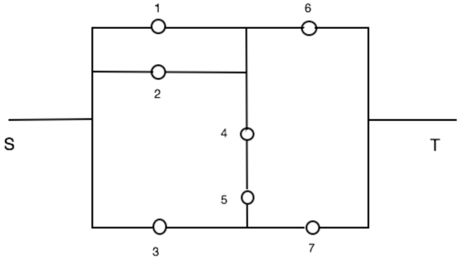
STK3405/4405 - 2015, problem 1 (a)



(a) Minimal path sets:

$$\{1,6\},\{2,6\},\{1,4,5,7\},\{2,4,5,7\},\{3,7\},\{3,4,5,6\}.$$

Minimal cut sets:

$$\{1,2,3\},\{1,2,4,7\},\{1,2,5,7\},\{3,4,6\},\{3,5,6\},\{6,7\}.$$

STK3405/4405 - 2015, problem 1 (b)

By using the multiplication method based on either the minimal path sets or the minimal cut sets:

$$\phi(\boldsymbol{X}) = \coprod_{j=1}^{6} \prod_{i \in P_j} X_i = \prod_{j=1}^{6} \coprod_{i \in K_j} X_i,$$

we get $2^6 - 1 = 63$ terms before simplification.

By using total state space enumeration:

$$h(\boldsymbol{p}) = \sum_{\boldsymbol{X} \in \{0,1\}^7} \phi(\boldsymbol{x}) P(\boldsymbol{X} = \boldsymbol{x}),$$

we get $2^7 - 1 = 127$ terms before simplification (since there are 7 components).

STK3405/4405 - 2015, problem 1 (c)

Find the reliability of this system as a function of the component reliabilities $p_1, \ldots, p_7$.

By factoring with respect to the bridge consisting of the series connection of the components 4 and 5, we get:

$$h(\boldsymbol{p}) =$$

$$p_4 p_5[((p_1 + p_2 - p_1 p_2) + p_3 - (p_1 + p_2 - p_1 p_2)p_3)$$

$$\cdot (p_6 + p_7 - p_6 p_7)]$$

$$+ (1 - p_4 p_5)[(p_1 + p_2 - p_1 p_2)p_6 + p_3 p_7$$

$$- (p_1 + p_2 - p_1 p_2)p_6 \cdot p_3 p_7]$$

STK3405/4405 - 2015, problem 1 (d)

What is the reliability importance of component 4 according to the Birnbaum measure?

$$
\begin{aligned}
I_B^{(4)} &= \partial h(\boldsymbol{p})/\partial p_4 \\
&\quad p_5[((p_1 + p_2 - p_1 p_2) + p_3 - (p_1 + p_2 - p_1 p_2)p_3) \\
&\quad \cdot (p_6 + p_7 - p_6 p_7)] \\
&\quad - p_5[(p_1 + p_2 - p_1 p_2)p_6 + p_3 p_7 \\
&\quad - (p_1 + p_2 - p_1 p_2)p_6 \cdot p_3 p_7]
\end{aligned}
$$

STK3405/4405 - 2015, problem 1 (e)

What is the corresponding structural importance of component 4?

By letting $p_i = 1/2$ for $i = 1, 2, 3, 5, 6, 7$ in $I_B^{(4)}$ we get:

$$
\begin{aligned}
J_B^{(4)} &= (1/2)[((3/4) + (1/2) - (3/4) \cdot (1/2))(3/4)] \\
&\quad - (1/2)[(3/4)(1/2) + (1/2)(1/2) - (3/4)(1/2) \cdot (1/2)(1/2)] \\
&= 4/64 = 1/16.
\end{aligned}
$$

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Alternatively, we have:

$$J_B^{(4)} = \frac{1}{2^{7-1}} \cdot \text{The number of critical vectors for component 4.}$$

We have the following critical vectors for component 4:

$$(1,0,0,\cdot,1,0,1), \quad (0,1,0,\cdot,1,0,1),$$
$$(1,1,0,\cdot,1,0,1), \quad (0,0,1,\cdot,1,1,0),$$

Hence,

$$J_B^{(4)} = \frac{4}{64} = \frac{1}{16}.$$

NOTE: The critical vectors correspond to the following path sets:

$$\{1,4,5,7\}, \quad \{2,4,5,7\},$$
$$\{3,4,5,6\}, \quad \{1,2,4,5,7\},$$

where the first three are minimal path sets containing component 4.

STK3405/4405 - 2015, problem 1 (f)

What is the structural importance of component 1?

$$I_B^{(1)} = \partial h(\boldsymbol{p})/\partial p_1$$

$$p_4 p_5[((1-p_2)-(1-p_2)p_3)\cdot(p_6+p_7-p_6p_7)]$$

$$+(1-p_4p_5)[(1-p_2)p_6-(1-p_2)p_6\cdot p_3p_7]$$

$$= p_4p_5[(1-p_2)(1-p_3)(p_6+p_7-p_6p_7)]$$

$$+(1-p_4p_5)[(1-p_2)p_6(1-p_3p_7)]$$

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Hence, the structural importance of component 1 is:

$$J_B^{(1)} = (1/4)[(1/2)(1/2)(3/4)] + (3/4)[(1/2)(1/2)(3/4)]$$
$$= (3/64) + (9/64) = 12/64 = 3/16.$$

We observe that $J_B^{(1)} > J_B^{(4)}$. Thus, component 1 and component 2 are structurally more important than the bridge components 4 and component 5.

STK3405/4405 - 2015, problem 2

In this exercise you may use that if $X_1, \ldots, X_n$ are associated, binary, random variables, then:

$$E[\prod_{i=1}^{n} X_i] \geq \prod_{i=1}^{n} E[X_i]$$

$$E[\coprod_{i=1}^{n} X_i] \leq \coprod_{i=1}^{n} E[X_i]$$

STK3405/4405 - 2015, problem 2 (a)

Show that independent random variables are associated.

PROOF: Let $T_1, \ldots, T_n$ be *independent* random variables. We shall prove that they are also *associated*. The proof is by induction on n. The result obviously holds for $n = 1$ by Theorem 6.1.4 (*ii*).

Assume that the theorem holds for $n = m - 1$. That is, $\{T_1, \ldots, T_{m-1}\}$ is a set of associated random variables. Moreover, by Theorem 6.1.4 (*ii*), $\{T_m\}$ is associated as well.

By the assumption, these two sets are independent. Hence, it follows from Theorem 6.1.4 (*iv*) that their union $\{T_1, \ldots, T_{m-1}, T_m\}$ is a set of associated random variables.

Thus, the result is proved by induction.

STK3405/4405 - 2015, problem 2 (b)

Let $X_1, \ldots, X_n$ be the associated component state variables of a non-trivial binary monotone structure (C, ϕ) with component reliabilities $p_1, \ldots, p_n$. Show that:

$$\prod_{i=1}^{n} p_i \leq P[\phi(\boldsymbol{x}) = 1] \leq \coprod_{i=1}^{n} p_i.$$

PROOF: Since (C, ϕ) is *non-trivial*, we know that:

$$\prod_{i=1}^{n} X_i \leq \phi(\boldsymbol{X}) \leq \coprod_{i=1}^{n} X_i$$

Hence, we get

$$\prod_{i=1}^{n} p_i = \prod_{i=1}^{n} E[X_i] \leq E[\prod_{i=1}^{n} X_i] \leq P[\phi(\boldsymbol{x}) = 1]$$

$$\leq E[\coprod_{i=1}^{n} X_i] \leq \coprod_{i=1}^{n} E[X_i] = \coprod_{i=1}^{n} p_i.$$

STK3405/4405 - 2015, problem 2 (c)

Show that:

$$\prod_{j=1}^{k} P(\kappa_j(\boldsymbol{X}^{K_j}) = 1) \leq P[\phi(\boldsymbol{x}) = 1] \leq \prod_{j=1}^{p} P(\rho_j(\boldsymbol{X}^{P_j}) = 1).$$

PROOF: It follows from Theorem 6.1.4 (*iii*) that the minimal path series structures, and the minimal cut parallel structures, are associated. Hence, we get that:

$$\begin{aligned}\prod_{j=1}^{k} P(\kappa_j(\boldsymbol{X}^{K_j}) = 1) \leq E[\prod_{j=1}^{k} \kappa_j(\boldsymbol{X}^{K_j})] &= P[\phi(\boldsymbol{x}) = 1] \\ &= E[\prod_{j=1}^{p} \rho_j(\boldsymbol{X}^{P_j})] \leq \prod_{j=1}^{p} P(\rho_j(\boldsymbol{X}^{P_j}) = 1),\end{aligned}$$

Unfortunately, these bounds are not explicit.

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Asssume in addition that $X_1, \ldots, X_n$ are independent. Show that:

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq P[\phi(\boldsymbol{x}) = 1] \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i.$$

PROOF: For independent components, the lower bound is equal to the one in (c) because:

$$P(\kappa_j(\boldsymbol{X}^{K_j}) = 1) = E[\coprod_{i \in K_j} X_i] = \coprod_{i \in K_j} p_i.$$

Similarly the upper bound is equal to the one in (c) because:

$$P(\rho_j(\boldsymbol{X}^{K_j}) = 1) = E[\prod_{i \in P_j} X_i] = \prod_{i \in P_j} p_i.$$

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Assume in addition that the binary monotone system (C, ϕ) has at least two minimal cut sets that overlap and that $0 < p_i < 1$, $i = 1, \ldots, n$. Show then that we have:

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i < P[\phi(\boldsymbol{x}) = 1].$$

PROOF: Assume without loss of generality that $K_1 \cap K_2 \neq \emptyset$.

Since we have assumed that $0 < p_i < 1$, $i = 1, \ldots, n$, it follows that $\coprod_{i \in K_1} X_i$ and $\prod_{j=2}^{k} \coprod_{i \in K_j} X_i$ are *dependent*. Thus, we must have:

$$\mathrm{Cov}(\coprod_{i \in K_1} X_i, \prod_{j=2}^{k} \coprod_{i \in K_j} X_i) > 0.$$

Hence, it follows that:

$$E[\prod_{j=1}^{k} \coprod_{i \in K_j} X_i] = E[\coprod_{i \in K_1} X_i \cdot \prod_{j=2}^{k} \coprod_{i \in K_j} X_i] > E[\coprod_{i \in K_1} X_i] \cdot E[\prod_{j=2}^{k} \coprod_{i \in K_j} X_i].$$

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Thus, we have:

$$P[\phi(\boldsymbol{x}) = 1] = E[\prod_{j=1}^{k} \coprod_{i \in K_j} X_i]$$

$$> E[\coprod_{i \in K_1} X_i] \cdot E[\prod_{j=2}^{k} \coprod_{i \in K_j} X_i] \geq E[\coprod_{i \in K_1} X_i] \cdot \prod_{j=2}^{k} E[\coprod_{i \in K_j} X_i]$$

$$= \prod_{j=1}^{k} E[\coprod_{i \in K_j} X_i] = \prod_{j=1}^{k} \coprod_{i \in K_j} p_i.$$

STK3405/4405 - 2015, problem 2 (f)

Show that for a specific k-out-of-n system with $p_i = p$, $i = 1, \ldots, n$ that the lower bound in (d) can be poorer than the lower bound in (b).

PROOF: Consider a 3-out-of-4 system med $p_i = p$, $i = 1, 2, 3, 4$. Then the lower bound in (d) becomes:

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i = \prod_{j=1}^{6} (p + p - p^2) = p^6 (2 - p)^6.$$

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We then compare this to the lower bound in (b), i.e, p^4, and look for some $p \in (0, 1)$ such that:

$$p^6(2 - p)^6 < p^4$$

or equivalently:

$$p^2(2 - p)^6 < 1$$

By choosing $p = 1/10$ we get that:

$$p^2(2 - p)^6 = (1/10)^2(2 - 1/10)^6 < 2^6/10^2 = 64/100 < 1.$$

Thus, for this value of p the lower bound in (d) is poorer than the lower bound in (b).