Exam STK3405/4405 - 2016

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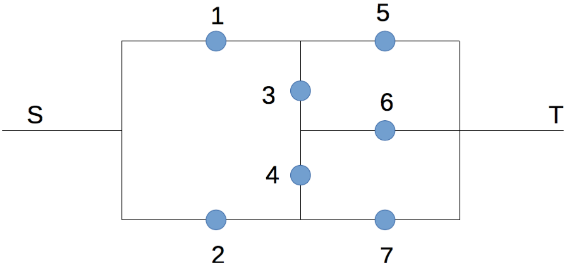
STK3405/4405 - 2016, problem 1 (a)



(a) Minimal path sets:

$$\{1,5\},\{1,3,6\},\{1,3,4,7\},\{2,7\},\{2,4,6\},\{2,3,4,5\}.$$

Minimal cut sets:

$$\{1,2\},\{2,3,5\},\{1,4,7\},\{5,6,7\},\{3,4,5,7\},\{2,4,5,6\},\{1,3,6,7\}.$$

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Since there are fewer minimal path sets than minimal cut sets, it is best to use the minimal path sets in the multiplication method.

By doing this, we will in the best possible case get $2^6 - 1 = 63$ terms by using the multiplication method before simplification.

By total state enumeration, we get $2^7 - 1 = 127$ terms (since there are 7 components).

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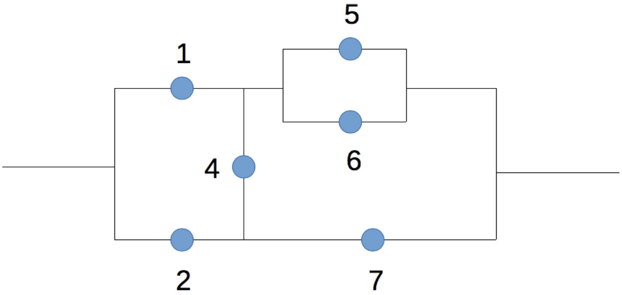
Find the reliability of this system as a function of the component reliabilities $p_1, \ldots, p_7$ using the factoring algorithm.

SOLUTION: We pivot with respect to component 3.

If component 3 *is functioning:*



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By parallel-reducing components 5 and 6, we see that this is a bridge structure. Hence, we can find the reliability by pivoting with respect to the bridge component, i.e. component 4.

$$
\begin{aligned}
h(1_3, \boldsymbol{p}) &= p_4(p_1 \amalg p_2)(p_5 \amalg p_6 \amalg p_7) \\
&+ (1 - p_4)[(p_1(p_5 \amalg p_6)) \amalg p_2 p_7] \\
&= p_4(p_1 + p_2 - p_1 p_2) \\
&\cdot (p_5 + p_6 + p_7 - p_5 p_6 - p_5 p_7 - p_6 p_7 + p_5 p_6 p_7) \\
&+ (1 - p_4)[p_1(p_5 + p_6 - p_5 p_6) + p_2 p_7 \\
&- p_1(p_5 + p_6 - p_5 p_6) p_2 p_7]
\end{aligned}
$$

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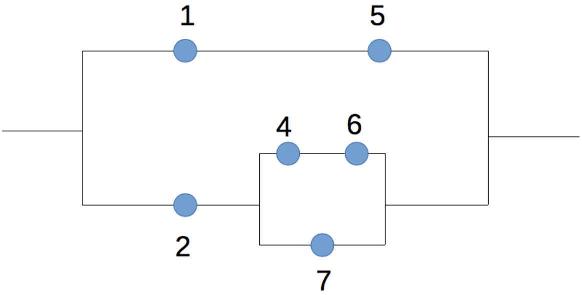
If component 3 *is not functioning:*



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This is an s-p system where:

$$\begin{aligned} h(0_3, \boldsymbol{p}) &= (p_1 p_5) \amalg [p_2(p_4 p_6 \amalg p_7)] \\ &= (p_1 p_5) \amalg [p_2(p_4 p_6 + p_7 - p_4 p_6 p_7)] \\ &= p_1 p_5 + p_2(p_4 p_6 + p_7 - p_4 p_6 p_7) \\ &- p_1 p_2 p_5(p_4 p_6 + p_7 - p_4 p_6 p_7) \end{aligned}$$

Finally, the reliability function is:

$$h(\boldsymbol{p}) = p_3 h(1_3, \boldsymbol{p}) + (1 - p_3) h(0_3, \boldsymbol{p})$$

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What is reliability importance of component 3 according to the Birnbaum measure?

SOLUTION: We know that:

$$I_B^{(3)} = \frac{\partial h(\boldsymbol{p})}{\partial p_3} = h(1_3, \boldsymbol{p}) - h(0_3, \boldsymbol{p}).$$

Hence, by using (c), we see that:

$$\begin{aligned}
I_B^{(3)} &= p_4(p_1 + p_2 - p_1 p_2)(p_5 + p_6 + p_7 - p_5 p_6 - p_5 p_7 - p_6 p_7 + p_5 p_6 p_7) \\
&+ (1 - p_4)\big[p_1(p_5 + p_6 - p_5 p_6) + p_2 p_7 - p_1(p_5 + p_6 - p_5 p_6)p_2 p_7\big] \\
&- \big[p_1 p_5 + p_2(p_4 p_6 + p_7 - p_4 p_6 p_7) - p_1 p_2 p_5(p_4 p_6 + p_7 - p_4 p_6 p_7)\big]
\end{aligned}$$

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What is the corresponding structural importance of component 3?

SOLUTION:

$$J_B^{(3)} = I_B^{(3)}\big|_{p_1=\cdots=p_7=1/2}$$

$$= \big[h(1_3, \boldsymbol{p}) - h(0_3, \boldsymbol{p})\big]\big|_{p_1=\cdots=p_7=1/2}$$

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Here we have:

$$\begin{aligned}
h(1_3, \boldsymbol{p})|p_i &= 1/2, i = 1, \ldots, 7 \\
&= (\frac{1}{2})(\frac{3}{4})[\frac{3}{4} + \frac{1}{2} - \frac{3}{8}] + (\frac{1}{2})[(\frac{1}{2})(\frac{3}{4}) + (\frac{1}{4}) - (\frac{1}{8})(\frac{3}{4})] \\
&= (\frac{3}{8})(\frac{7}{8}) + (\frac{1}{2})((\frac{5}{8}) - (\frac{3}{32})) = (\frac{21}{64}) + (\frac{17}{64}) = \frac{38}{64}
\end{aligned}$$

and:

$$\begin{aligned}
h(0_3, \boldsymbol{p})|p_i &= 1/2, i = 1, \ldots, 7 \\
&= (\frac{1}{4}) + (\frac{1}{2})((\frac{1}{4}) + (\frac{1}{2}) - (\frac{1}{8})) - (\frac{1}{8})((\frac{1}{4}) + (\frac{1}{2}) - (\frac{1}{8})) \\
&= (\frac{1}{4}) + (\frac{5}{16}) - (\frac{5}{64}) = (\frac{16}{64}) + (\frac{20}{64}) - (\frac{5}{64}) = \frac{31}{64}.
\end{aligned}$$

Hence,

$$J_B^{(3)} = \frac{38}{64} - \frac{31}{64} = \frac{7}{64} = \frac{7}{2^{7-1}}.$$

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Alternatively:

$$J_B^{(3)} = \frac{1}{2^{7-1}} \text{the number of critical vectors for component 3}$$

Since we have the following 7 critical vectors for component 3, the previous answer is verified:

$$\begin{array}{lll} (1,0,\cdot,0,0,1,0), & (1,0,\cdot,1,0,0,1), & (0,1,\cdot,1,1,0,0), \\ (1,0,\cdot,0,0,1,1), & (1,0,\cdot,1,0,1,0), & (1,0,\cdot,1,0,1,1), \\ (1,1,\cdot,0,0,1,0) & & \end{array}$$

NOTE: These critical vectors correspond to the following path sets:

$$\begin{array}{lll} \{1,3,6\}, & \{1,3,4,7\}, & \{2,3,4,5\}, \\ \{1,3,6,7\}, & \{1,3,4,6\}, & \{1,3,4,6,7\}, \\ \{1,2,3,6\}, & & \end{array}$$

where the first three are minimal path sets containing component 3.

STK3405/4405 - 2016, problem 2 (a)

Consider a binary monotone system (C, ϕ) of n components which are not repaired, and where the state processes of the components $\{X_i(t), t \geq 0\}_{i=1,\ldots,n}$ are independent.

Birnbaum (1969) suggested the following measure for the reliability importance of the ith component at time t:

$$I_B^{(i)}(t) = \frac{\partial h(\boldsymbol{p}(t))}{\partial p_i(t)}, i = 1, \ldots, n$$

Prove that:

$$I_B^{(i)}(t) = P[(\cdot_i, \boldsymbol{X}(t)) \text{ is a critical vector for component } i]$$

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SOLUTION: By pivoting on component i, we have:

$$h(\boldsymbol{p}(t)) = p_i(t) \cdot h(1_i, \boldsymbol{p}(t)) + (1 - p_i(t)) \cdot h(0_i, \boldsymbol{p}(t)).$$

Thus, the partial derivative of h with respect to $p_i(t)$ is given by:

$$\begin{aligned}
\frac{\partial h(\boldsymbol{p}(t))}{\partial p_i(t)} &= h(1_i, \boldsymbol{p}(t)) - h(0_i, \boldsymbol{p}(t)) \\
&= E[\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t))] \\
&= P[\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t)) = 1] \\
&= P[(\cdot_i, \boldsymbol{X}(t)) \text{ is a critical vector for component } i]
\end{aligned}$$

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The Barlow and Proschan (1975) measure of reliability importance of the *i*th component is defined by:

$$I_{B-P}^{(i)} = P(\text{Component } i \text{ fails at the same time as the system}).$$

Show that:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt,$$

where $f_i(t)$ is the probability density of the lifetime of the *i*th component.

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SOLUTION: From the definitions of the Barlow-Proschan measure and the Birnbaum measure, it follows that:

$$
\begin{aligned}
I_{B-P}^{(i)} &= P(\text{Component } i \text{ fails at the same time as the system}) \\
&= \int_0^\infty P(\text{Component } i \text{ is critical at time } t) \cdot f_i(t)dt \\
&= \int_0^\infty I_B^{(i)}(t) f_i(t)dt.
\end{aligned}
$$

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The Vesely and Fussel (1975) measure is given by:

$$I_{V-F}^{(i)}(t) = P[X_i(t) = 0 | \phi(X(t)) = 0]$$

The Vesely-Fussel measure was intended as a *diagnostic measure*, not as an *importance measure*.

Given that the system is failed at a given point of time t, which components are most likely to being failed.

Thus, the Vesely-Fussel measure is useful when the system is investigated after a failure in order to identify failed components.

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NOTE: The Vesely-Fussel measure has the following properties:

- In a parallel system we know that all the components are failed when the system is failed. Thus, in a parallel system we have $I^{(1)}_{V-F}(t) = \cdots = I^{(n)}_{V-F}(t)$.
- $I^{(1)}_{V-F}(t)$ may be positive even if component i fails after the system.
- $I^{(1)}_{V-F}(t)$ may be positive even if component i is irrelevant.

These properties indicate that the Vesely-Fussel measure may not be suitable as a measure of importance.

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Assume that the ith component is *irrelevant* for the binary monotone system (C, ϕ). Thus, we have:

$$\phi(1_i, \boldsymbol{X}(t)) = \phi(0_i, \boldsymbol{X}(t)), \text{ for all } (\cdot, \boldsymbol{X}(t)).$$

From this it follows that:

$$I_B^{(i)}(t) = P[\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t)) = 1] = 0.$$

Hence, using (b) we also get that:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt = 0.$$

However, as already explained, we may still have $I_{V-F}^{(i)}(t) > 0$.

STK3405/4405 - 2016, problem 3

We skip this as the solutions to this problem are essentially covered by the previous problems in this presentation.