Exam STK3405/4405 - 2018

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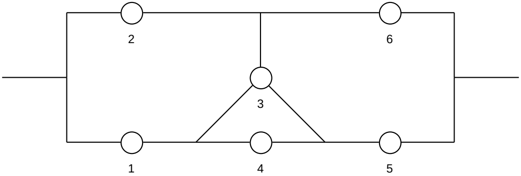
Problem 1

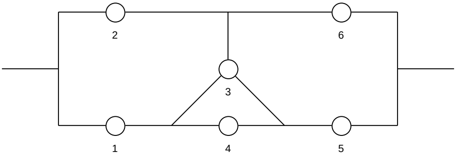


Let (C, ϕ) be the network system shown above, where $C = \{1, 2, \ldots, 6\}$. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_6)$ denote the vector of component state variables, and assume that $X_1, X_2, \ldots, X_6$ are stochastically independent.

NOTE: Component 3 is a *node*, not an *edge* in the network!

Let $\boldsymbol{p} = (p_1, p_2, \ldots, p_6)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, \ldots, 6$. We assume that $0 < p_i < 1$ for $i = 1, 2, \ldots, 6$.

Problem 1a



(a) Find the minimal path and cut sets of the system.

SOLUTION:

Minimal path sets: $P_1 = \{1, 4, 5\}$, $P_2 = \{2, 6\}$, $P_3 = \{1, 3, 6\}$, $P_4 = \{2, 3, 5\}$, $P_5 = \{1, 3, 5\}$

Minimal cut sets: $K_1 = \{1, 2\}$, $K_2 = \{1, 3, 6\}$, $K_3 = \{2, 3, 4\}$, $K_4 = \{2, 3, 5\}$, $K_5 = \{3, 4, 6\}$, $K_6 = \{5, 6\}$

Problem 1b

(b) Use the result in a) to find an expression for the structure function of the system, and explain briefly how this can be used to find the system reliability. A detailed calculation is *not required*.

SOLUTION: The structure function can be expressed either as a series connection of the minimal cut parallel structures:

$$\phi(\boldsymbol{X}) = \prod_{j=1}^{6} \coprod_{i \in K_j} X_i$$

or as a parallel connection of the minimal path series structures:

$$\phi(\boldsymbol{X}) = \coprod_{j=1}^{5} \prod_{i \in P_j} X_i$$

Problem 1b (cont.)

The system reliability, $h(\boldsymbol{p})$, is defined as:

$$h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})] = \mathrm{E}[\coprod_{j=1}^{6} \prod_{i \in K_j} X_i] = \mathrm{E}[\prod_{j=1}^{5} \coprod_{i \in P_j} X_i]$$

The system reliability can be found by expanding either of the expressions for the structure function into a sum of products of the component state variables:

$$\phi(\boldsymbol{X}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} X_i$$

We can then, in principle, find the reliability of the system as:

$$\mathrm{E}[\phi(\boldsymbol{X})] = \sum_{A \subseteq C} \delta(A)\, \mathrm{E}[\prod_{i \in A} X_i] = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} p_i$$

where the last equality follows since we have assumed that the component state variables are *independent*.

Problem 1c

(c) Use the factoring algorithm to derive the reliability of the system in a different way from the one in b).

SOLUTION: By pivoting with respect to component 3, we see that the reliability $h(\boldsymbol{p})$ is:

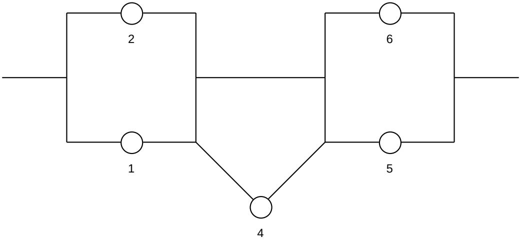
$$h(\boldsymbol{p}) = p_3 h(1_3, \boldsymbol{p}) + (1 - p_3) h(0_3, \boldsymbol{p})$$

In the following we consider $h(1_3, \boldsymbol{p})$ and $h(0_3, \boldsymbol{p})$ separately where:

- $h(1_3, \boldsymbol{p})$ corresponds to the case where component 3 is *functioning*
- $h(0_3, \boldsymbol{p})$ corresponds to the case where component 3 is *failed*

Problem 1c (cont.)

Component 3 **is functioning:**
In this case, the system becomes a series connection of two parallel connections (1 and 2 in parallel), (5 and 6 in parallel). Component 4 becomes irrelevant.



Hence, we get:

$$h(1_3, \boldsymbol{p}) = (p_1 \amalg p_2)(p_5 \amalg p_6) = (p_1 + p_2 - p_1 p_2)(p_5 + p_6 - p_5 p_6).$$

Problem 1c (cont.)

Component 3 is failed:
In this case, the system becomes a parallel connection of two series connections, (2 and 6 in series), (1, 4 and 5 in series), so the reliability function in this case is:

$$h(0_3, \boldsymbol{p}) = (p_2 p_6) \amalg (p_1 p_4 p_5) = p_2 p_6 + p_1 p_4 p_5 - p_1 p_2 p_4 p_5 p_6.$$

Hence, the reliability function is:

$$\begin{aligned} h(\boldsymbol{p}) &= p_3 h(1_3, \boldsymbol{p}) + (1 - p_3) h(0_3, \boldsymbol{p}) \\ &= p_3(p_1 + p_2 - p_1 p_2)(p_5 + p_6 - p_5 p_6) \\ &+ (1 - p_3)(p_2 p_6 + p_1 p_4 p_5 - p_1 p_2 p_4 p_5 p_6). \end{aligned}$$

NOTE: If you pivot with respect to any of the other components, you will have to perform one additional pivot.

Problem 1d

(d) What is the definition of the Birnbaum measure for the reliability importance of a component?

SOLUTION: Let (C, ϕ) be a binary monotone system, and let $i \in C$. Moreover, let $\boldsymbol{X}$ denote the vector of component state variables.

The Birnbaum measure for the reliability importance of component i, denoted $I_B^{(i)}$, is defined as:

$$
\begin{aligned}
I_B^{(i)} &= P(\text{Component } i \text{ is critical for the system}) \\
&= P(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1) \\
&= \mathrm{E}[\phi(1_i, \boldsymbol{X})] - \mathrm{E}[\phi(0_i, \boldsymbol{X})]
\end{aligned}
$$

When the component state variables are independent, we get that:

$$I_B^{(i)} = h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p})$$

where $\boldsymbol{p}$ denotes the vector of component reliabilities.

Problem 1e

(e) What is the reliability importance of the component 3 according to the Birnbaum measure?

SOLUTION: From (d) we have that:

$$I_B^{(3)} = h(1_3, \boldsymbol{p}) - h(0_3, \boldsymbol{p}).$$

Hence, by using the results from (c), we get that:

$$I_B^{(3)} = (p_1 + p_2 - p_1 p_2)(p_5 + p_6 - p_5 p_6)$$
$$- (p_2 p_6 + p_1 p_4 p_5 - p_1 p_2 p_4 p_5 p_6)$$

Problem 1e (cont.)

(e) Use this to find the structural importance of component 3.

SOLUTION: Using the expression for $I_B^{(3)}$ from the previous slide we get:

$$J_B^{(3)} = I_B^{(3)}\Big|_{p_1=\cdots=p_6=\frac{1}{2}} = \left[h(1_3, \boldsymbol{p}) - h(0_3, \boldsymbol{p})\right]\Big|_{p_1=\cdots=p_6=\frac{1}{2}}$$

$$= \cdots = \frac{7}{32}.$$

Alternatively:

$$J_B^{(3)} = \tfrac{1}{32} \cdot \left[\text{The number of critical path sets for component 3}\right]$$

Since we have the following 7 critical path sets for component 3, the previous answer is verified:

$$\{1,3,5\}, \{1,3,6\}, \{2,3,5\}, \{1,2,3,5\}, \{1,3,5,6\}, \{1,3,4,6\}, \{2,3,4,5\}.$$

Problem 1f

(f) Assume that $p_i = p$ for $i = 1, \ldots, 6$. What can you say about the reliability importance of the other components?

SOLUTION: By symmetry the reliability importance of components 1 and 5 must be the same. Moreover, by symmetry the reliability importance of components 2 and 6 must be the same.

Explicit expressions for the reliability importance measures can be computed in a similar way as in (e). We skip the details here.

Problem 2a

Consider a binary monotone system (C, ϕ), where $C = \{1, 2, 3\}$ and where the structure function ϕ is given by:

$$\phi(\boldsymbol{X}) = \mathrm{I}(\sum_{i=1}^{3} X_i \geq 2).$$

Here $\boldsymbol{X} = (X_1, X_2, X_3)$ denotes the vector of component state variables and $\mathrm{I}(\cdot)$ denotes the indicator function.

(a) Show that the structure function ϕ can be written as:

$$\phi(\boldsymbol{X}) = X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3.$$

Problem 2a (cont.)

SOLUTION: By using pivotal decomposition we get:

$$\begin{aligned}
\phi(\boldsymbol{X}) &= X_1\phi(1_1, \boldsymbol{X}) + (1 - X_1)\phi(0_1, \boldsymbol{X}) \\
&= X_1 \cdot \mathrm{I}(\sum_{i=2}^{3} X_i \geq 1) + (1 - X_1) \cdot \mathrm{I}(\sum_{i=2}^{3} X_i \geq 2) \\
&= X_1 \cdot (X_2 \amalg X_3) + (1 - X_1) \cdot X_2 X_3 \\
&= X_1 \cdot (X_2 + X_3 - X_2 X_3) + (1 - X_1) \cdot X_2 X_3 \\
&= X_1 X_2 + X_1 X_3 - X_1 X_2 X_3 + X_2 X_3 - X_1 X_2 X_3 \\
&= X_1 X_2 + X_1 X_3 + X_2 X_3 - 2 X_1 X_2 X_3
\end{aligned}$$

Problem 2b

In the following we assume that:

$$X_i = Y_0 \cdot Y_i, \quad i = 1, 2, 3,$$

where Y_0, Y_1, Y_2, Y_3 are independent binary stochastic variables and:

$$P(Y_0 = 1) = \theta, \quad P(Y_1 = 1) = P(Y_2 = 1) = P(Y_3 = 1) = q,$$

where $0 < \theta < 1$ and $0 < q < 1$.

(b) Explain why this implies that X_1, X_2, X_3 are associated stochastic variables.

SOLUTION: Since X_1, X_2, X_3 are non-decreasing functions of the independent variables Y_0, Y_1, Y_2, Y_3, it follows that X_1, X_2, X_3 are associated stochastic variables.

Problem 2c

(c) Show that:

$$h = h(\theta, q) = \mathrm{E}[\phi(\boldsymbol{X})] = \theta q^2(3 - 2q).$$

SOLUTION: By conditioning on Y_0 we get:

$$\begin{aligned} h(\theta, q) &= \mathrm{E}[\phi(\boldsymbol{X})] \\ &= P(Y_0 = 1) \cdot \mathrm{E}[\phi(\boldsymbol{X}) | Y_0 = 1] + P(Y_0 = 0) \cdot \mathrm{E}[\phi(\boldsymbol{X}) | Y_0 = 0] \\ &= \theta \cdot \mathrm{E}[Y_1 Y_2 + Y_1 Y_3 + Y_2 Y_3 - 2 Y_1 Y_2 Y_3] + (1 - \theta) \cdot 0 \\ &= \theta(3q^2 - 2q^3) = \theta q^2(3 - 2q). \end{aligned}$$

Problem 2d

Assume that we ignore the dependence between the X_is, and instead computes the system reliability as if X_1, X_2, X_3 are independent and:

$$P(X_i = 1) = \theta q, \quad i = 1, 2, 3.$$

Let $\tilde{h}$ denote the system reliability we then get.

(d) Show that:

$$\tilde{h} = \tilde{h}(\theta, q) = \theta^2 q^2 (3 - 2\theta q).$$

SOLUTION: Assuming that X_1, X_2, X_3 are independent we get:

$$\begin{aligned}\tilde{h} &= \mathrm{E}[X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3] \\ &= (\theta \cdot q)^2 + (\theta \cdot q)^2 + (\theta \cdot q)^2 - 2(\theta \cdot q)^3 \\ &= \theta^2 q^2 (3 - 2\theta q).\end{aligned}$$

Problem 2e

(e) Assume that $\theta = \frac{1}{2}$. Show that we then have $\tilde{h} < h$ for all $0 < q < 1$.

SOLUTION: We introduce $d(\theta, q) = h(\theta, q) - \tilde{h}(\theta, q)$ given by:

$$\begin{aligned} d(\theta, q) = h(\theta, q) - \tilde{h}(\theta, q) &= \theta q^2(3 - 2q) - \theta^2 q^2(3 - 2\theta q) \\ &= \theta q^2[3 - 2q - 3\theta + 2\theta^2 q] \\ &= \theta q^2[3(1 - \theta) - 2q(1 - \theta^2)] \\ &= \theta(1 - \theta)q^2[3 - 2q(1 + \theta)] \end{aligned}$$

If $\theta = \frac{1}{2}$ we get for all $0 < q < 1$ that:

$$d(\theta, q) = (\frac{1}{4})q^2[3 - 2q(\frac{3}{2})] = (\frac{3}{4})q^2[1 - q] > 0.$$

Hence, $\tilde{h} < h$ for all $0 < q < 1$. Thus, in this case we always underestimate the system reliability if we ignore the dependence.

Problem 2f

(f) Assume instead that $\theta = \frac{3}{4}$. What can you say about the relationship between $\tilde{h}$ and h in this case?

SOLUTION: If $\theta = \frac{3}{4}$ we get that:

$$d(\theta, q) = (\frac{3}{16})q^2[3 - 2q(\frac{7}{4})] = (\frac{3}{32})q^2[6 - 7q]$$

Hence, $d(\theta, q) > 0$ if and only if $7q < 6$. Thus, we conclude that $\tilde{h} < h$ if and only if $q < \frac{6}{7}$. Conversely, $\tilde{h} > h$ if and only if $q > \frac{6}{7}$.

We observe that in this case we always underestimate the system reliability if we ignore the dependence when $q < \frac{6}{7}$, while we overestimate the system reliability if we ignore the dependence when $q > \frac{6}{7}$.

Problem 3a

Let (C, ϕ) be a binary monotone system, and let $\boldsymbol{X}$ denote the vector of component state variables. In this problem we consider how the system reliability $h = P(\phi(\boldsymbol{X}) = 1)$ can be estimated using Monte Carlo simulation. The simplest Monte Carlo estimate is:

$$\hat{h}_{MC} = \frac{1}{N} \sum_{r=1}^{N} \phi(\boldsymbol{X}_r),$$

where $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ are data generated from the distribution of $\boldsymbol{X}$.

In order to improve this estimate we let $S = S(\boldsymbol{X})$ be a stochastic variable with values in the set $\{s_1, \ldots, s_k\}$. We assume that the distribution of S is known, and introduce:

$$\theta_j = E[\phi | S = s_j], \quad j = 1, \ldots, k.$$

Problem 3a (cont.)

We then use Monte Carlo simulation in order to estimate $\theta_1, \ldots, \theta_k$, and generate data from the conditional distribution of $\boldsymbol{X}$ given S. We let $\{\boldsymbol{X}_{r,j} : r = 1, \ldots, N_j\}$ denote the vectors generated from the distribution of $\boldsymbol{X}$ given that $S = s_j$, $j = 1, \ldots, k$, and get the following estimates:

$$\hat{\theta}_j = \frac{1}{N_j} \sum_{r=1}^{N_j} \phi(\boldsymbol{X}_{r,j}), \quad j = 1, \ldots, k.$$

These estimates are then combined into the following estimate of the system reliability:

$$\hat{h}_{CMC} = \sum_{j=1}^{k} \hat{\theta}_j P(S = s_j).$$

Problem 3a (cont.)

(a) Show that $E[\hat{h}_{CMC}] = h$ and that the variance of the estimate is given by:

$$\mathrm{Var}(\hat{h}_{CMC}) = \sum_{j=1}^{k} \frac{1}{N_j} \mathrm{Var}(\phi | S = s_j)[P(S = s_j)]^2$$

SOLUTION: We first note that the variances of the estimates $\hat{\theta}_1, \ldots, \hat{\theta}_k$ are given by:

$$\mathrm{Var}(\hat{\theta}_j) = \frac{1}{N_j^2} \sum_{r=1}^{N_j} \mathrm{Var}(\phi | S = s_j) = \frac{1}{N_j} \mathrm{Var}(\phi | S = s_j), \quad j = 1, \ldots, k.$$

Inserting this into the variance of $\hat{h}_{CMC}$ we get:

$$\mathrm{Var}(\hat{h}_{CMC}) = \mathrm{Var}[\sum_{j=1}^{k} \hat{\theta}_j P(S = s_j)] = \sum_{j=1}^{k} \mathrm{Var}(\hat{\theta}_j)[P(S = s_j)]^2$$

$$= \sum_{j=1}^{k} \frac{1}{N_j} \mathrm{Var}(\phi | S = s_j)[P(S = s_j)]^2.$$

Problem 3b

(b) Assume that $N_j \approx N \cdot P(S = s_j)$, $j = 1, \ldots, k$. Show that we then have:

$$\mathrm{Var}(\hat{h}_{CMC}) \approx \frac{1}{N}(\mathrm{Var}(\phi) - \mathrm{Var}[E(\phi|S)]),$$

and explain briefly why this implies that $\mathrm{Var}(\hat{h}_{CMC}) \leq \mathrm{Var}(\hat{h}_{MC})$.

SOLUTION: By inserting $N_j \approx N \cdot P(S = s_j)$, $j = 1, \ldots, k$ into the expression found in (a) we get:

$$Var(\hat{h}_{CMC}) \approx \sum_{j=1}^{k} \frac{1}{N \cdot P(S = s_j)} \mathrm{Var}(\phi|S = s_j)[P(S = s_j)]^2$$

$$= \frac{1}{N} \sum_{j=1}^{k} \mathrm{Var}(\phi|S = s_j) P(S = s_j) = \frac{1}{N} E[\mathrm{Var}(\phi|S)]$$

$$= \frac{1}{N}\big(\mathrm{Var}(\phi) - \mathrm{Var}[E(\phi|S)]\big),$$

Problem 3b (cont.)

Here we have used the formula:

$$\mathrm{Var}(\phi) = \mathrm{Var}[E(\phi|S)] + E[\mathrm{Var}(\phi|S)]$$

or equivalently:

$$E[\mathrm{Var}(\phi|S)] = \mathrm{Var}(\phi) - \mathrm{Var}[E(\phi|S)]$$

Since obviously $\mathrm{Var}[E(\phi|S)] \geq 0$, it follows that:

$$Var(\hat{h}_{CMC}) \approx \frac{1}{N}\big(\mathrm{Var}(\phi) - \mathrm{Var}[E(\phi|S)]\big) \leq \frac{1}{N}\mathrm{Var}(\phi) = \mathrm{Var}(\hat{h}_{MC})$$

Hence, we conclude that $\mathrm{Var}(\hat{h}_{CMC}) \leq \mathrm{Var}(\hat{h}_{MC})$.

Problem 3c

(c) What should one take into account when choosing S?

SOLUTION: The conditional estimate has smaller variance than the original Monte Carlo estimate provided that $\text{Var}[E(\phi \mid S)]$ is positive, which is a measure of how much information S contains relative to ϕ.

Thus, we should look for variables containing as much information about ϕ as possible.

Moreover, S needs to satisfy the following:

- S must have a distribution that can be computed analytically in polynomial time.
- The number of possible values of S, i.e., k, must be polynomially bounded by n.
- It must be possible to sample efficiently from the conditional distribution of $\boldsymbol{X}$ given S.

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