# Exam STK3405/4405 - 2019

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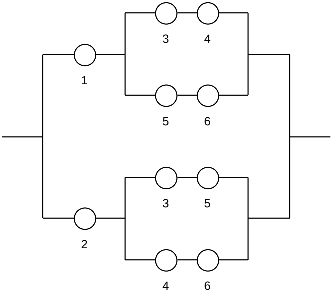
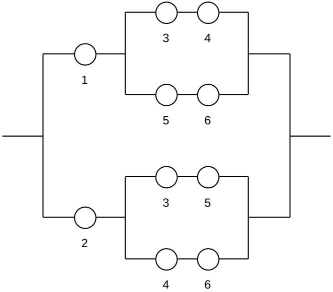
# Problem 1




Binary monotone system $(C, \phi)$ with component set of the system is $C = \{1, 2, \ldots, 6\}$. $\boldsymbol{X} = (X_1, X_2, \ldots, X_6)$ is the vector of component state variables, where $X_1, X_2, \ldots, X_6$ are stochastically independent.

Let $\boldsymbol{p} = (p_1, p_2, \ldots, p_6)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, \ldots, 6$.

# Problem 1a. Minimal path and cut sets of the system




**SOLUTION:**

Minimal path sets: $P_1 = \{1, 3, 4\}$, $P_2 = \{1, 5, 6\}$, $P_3 = \{2, 3, 5\}$, $P_4 = \{2, 4, 6\}$.

Minimal cut sets: $K_1 = \{1, 2\}$, $K_2 = \{3, 6\}$, $K_3 = \{4, 5\}$, $K_4 = \{1, 3, 4\}$, $K_5 = \{1, 5, 6\}$, $K_6 = \{2, 3, 5\}$, $K_7 = \{2, 4, 6\}$

# Problem 1b

(b) We let $h(\boldsymbol{p}) = P(\phi = 1)$ denote the reliability function of the system. Show that:

$$h(1_1, 1_2, \boldsymbol{p}) = E[\phi(1_1, 1_2, \boldsymbol{X})] = (p_3 \amalg p_6) \cdot (p_4 \amalg p_5),$$

$$h(1_1, 0_2, \boldsymbol{p}) = E[\phi(1_1, 0_2, \boldsymbol{X})] = (p_3 \cdot p_4) \amalg (p_5 \cdot p_6),$$

$$h(0_1, 1_2, \boldsymbol{p}) = E[\phi(0_1, 1_2, \boldsymbol{X})] = (p_3 \cdot p_5) \amalg (p_4 \cdot p_6),$$

and use this to find $h(\boldsymbol{p})$.

# Problem 1b (cont.)

**SOLUTION:**

CASE 1. $X_1 = 1$ and $X_2 = 1$.

We recall that the minimal cut sets of the system are:

$K_1 = \{1,2\}$, $K_2 = \{3,6\}$, $K_3 = \{4,5\}$, $K_4 = \{1,3,4\}$,
$K_5 = \{1,5,6\}$, $K_6 = \{2,3,5\}$, $K_7 = \{2,4,6\}$

In this case the minimal cut sets containing components 1 or 2 can be removed. The remaining minimal cut sets are:

$K_2 = \{3,6\}$ and $K_3 = \{4,5\}$.

This implies that:

$$\phi(1_1, 1_2, \boldsymbol{X}) = (X_3 \amalg X_6) \cdot (X_4 \amalg X_5),$$

and hence:

$$h(1_1, 1_2, \boldsymbol{p}) = (p_3 \amalg p_6) \cdot (p_4 \amalg p_5).$$

# Problem 1b (cont.)

CASE 2. $X_1 = 1$ and $X_2 = 0$.

We recall that the minimal path sets of the system are:

$$P_1 = \{1, 3, 4\},\ P_2 = \{1, 5, 6\},\ P_3 = \{2, 3, 5\},\ P_4 = \{2, 4, 6\}.$$

In this case the minimal paths containing component 2 can be removed, and thus, the resulting minimal path sets, where we have removed component 1 which we know is functioning, are:

$$P_1' = \{3, 4\} \text{ and } P_2' = \{5, 6\}.$$

This implies that:

$$\phi(1_1, 0_2, \boldsymbol{X}) = (X_3 \cdot X_4) \amalg (X_5 \cdot X_6),$$

and hence:

$$h(1_1, 0_2, \boldsymbol{p}) = (p_3 \cdot p_4) \amalg (p_5 \cdot p_6).$$

## Problem 1b (cont.)

CASE 3. $X_1 = 0$ and $X_2 = 1$.

We recall that the minimal path sets of the system are:

$$P_1 = \{1,3,4\},\ P_2 = \{1,5,6\},\ P_3 = \{2,3,5\},\ P_4 = \{2,4,6\}.$$

In this case the minimal paths containing component 1 can be removed, and thus, the resulting minimal path sets, where we have removed component 2 which we know is functioning, are:

$$P_3' = \{3,5\} \text{ and } P_4' = \{4,6\}.$$

This implies that:

$$\phi(0_1, 1_2, \boldsymbol{X}) = (X_3 \cdot X_5) \amalg (X_4 \cdot X_6),$$

and hence:

$$h(0_1, 1_2, \boldsymbol{p}) = (p_3 \cdot p_5) \amalg (p_4 \cdot p_6).$$

# Problem 1b (cont.)

Finally, since $\{1, 2\}$ is a minimal cut set, we have:

$$h(0_1, 0_2, \boldsymbol{p}) = \phi(0_1, 0_2, \boldsymbol{X}) = 0$$

Hence, combining all these results we get:

$$h(\boldsymbol{p}) = p_1 p_2 h(1_1, 1_2, \boldsymbol{p}) + p_1(1 - p_2) h(1_1, 0_2, \boldsymbol{p}) + (1 - p_1) p_2 h(0_1, 1_2, \boldsymbol{p})$$

$$= p_1 p_2 [(p_3 \amalg p_6) \cdot (p_4 \amalg p_5)]$$

$$+ p_1 (1 - p_2)[(p_3 \cdot p_4) \amalg (p_5 \cdot p_6)]$$

$$+ (1 - p_1) p_2 [(p_3 \cdot p_5) \amalg (p_4 \cdot p_6)]$$

# Problem 1c

In the remaining part of this problem we assume that all components have equal reliability $p$, i.e., $p_1 = \cdots = p_6 = p$. The reliability function can then be written as $h(p)$ instead of $h(\boldsymbol{p})$.

(c) Use the results from (b) to show that:

$$h(p) = p^4 \cdot (2 - p)^2 + 2p^3(1 - p)(2 - p^2)$$

In particular, show that:

$$h(\frac{1}{2}) = 23 \cdot (\frac{1}{2})^6$$

**SOLUTION:** We start by noting that:

$$s \amalg s = 1 - (1 - s)(1 - s) = 2s - s^2, \quad \text{for all } s.$$

# Problem 1c (cont.)

Inserting $p_1 = \cdots = p_6 = p$ into $h(\boldsymbol{p})$ we get:

$$
\begin{aligned}
h(p) &= p_1 p_2[(p_3 \amalg p_6) \cdot (p_4 \amalg p_5)] \\
&\quad + p_1(1 - p_2)[(p_3 \cdot p_4) \amalg (p_5 \cdot p_6)] + (1 - p_1)p_2[(p_3 \cdot p_5) \amalg (p_4 \cdot p_6)] \\
&= p^2[(2p - p^2) \cdot (2p - p^2)] + p(1 - p)[2p^2 - p^4] + (1 - p)p[2p^2 - p^4] \\
&= p^2 \cdot (2p - p^2)^2 + 2p(1 - p) \cdot (2p^2 - p^4) \\
&= p^4 \cdot (2 - p)^2 + 2p^3(1 - p)(2 - p^2).
\end{aligned}
$$

In particular we have:

$$
\begin{aligned}
h(\frac{1}{2}) &= (\frac{1}{2})^4 \cdot (\frac{3}{2})^2 + 2 \cdot (\frac{1}{2})^3 \cdot (\frac{1}{2}) \cdot (\frac{7}{4}) \\
&= [3^2 + 2 \cdot 7] \cdot (\frac{1}{2})^6 = 23 \cdot (\frac{1}{2})^6.
\end{aligned}
$$

# Problem 1d

(d) Let $S = \sum_{i=1}^{6} X_i$. Explain why the distribution of $S$ is given by:

$$P(S = s) = \binom{6}{s} p^s (1-p)^{6-s}, \quad s = 0, 1, \ldots, 6.$$

**SOLUTION:**
The random variable $S$ is the sum of the independent and identically binary variables $X_1, \ldots, X_6$.

Hence, $S \sim Bin(6, p)$, and thus $P(S = s)$ can be expressed as claimed.

# Problem 1e

(e) Show that:

$$h(p) = \sum_{s=0}^{6} b_s p^s (1-p)^{6-s}$$

where $b_s$ denotes the number of path sets (minimal and non-minimal) having exactly $s$ components, $s = 0, 1, \ldots, 6$.

**SOLUTION:**
We start by noting that:

$$b_s = \sum_{\{\boldsymbol{x} : \sum_{i=1}^{6} x_i = s\}} \phi(\boldsymbol{x}), \quad s = 0, 1, \ldots, 6.$$

# Problem 1e (cont.)

Moreover, the conditional distribution of $\boldsymbol{X}$ given $S$ is:

$$P(\boldsymbol{X} = \boldsymbol{x} \mid S = s) = \frac{p^{\sum_{i=1}^{6} x_i}(1-p)^{6-\sum_{i=1}^{6} x_i}}{\binom{6}{s} p^s (1-p)^{6-s}} = \frac{1}{\binom{6}{s}},$$

for all $\boldsymbol{x}$ such that $\sum_{i=1}^{6} x_i = s$, and zero otherwise.

From this it follows that:

$$\begin{aligned} E[\phi(\boldsymbol{X})|S = s] &= \sum_{\{\boldsymbol{X} : \sum_{i=1}^{6} x_i = s\}} \phi(\boldsymbol{x}) P(\boldsymbol{X} = \boldsymbol{x} \mid S = s) \\ &= \frac{1}{\binom{6}{s}} \sum_{\{\boldsymbol{X} : \sum_{i=1}^{6} x_i = s\}} \phi(\boldsymbol{x}) \; = \; \frac{b_s}{\binom{6}{s}} \end{aligned}$$

# Problem 1e (cont.)

Finally, the system reliability, $h$, expressed as a function of $p$, is given by:

$$
\begin{aligned}
h(p) &= E[\phi(\boldsymbol{X})] = \sum_{s=0}^{6} E[\phi(\boldsymbol{X}) | S = s] P(S = s) \\
&= \sum_{s=0}^{6} \frac{b_s}{\binom{6}{s}} \binom{6}{s} p^s (1-p)^{6-s} = \sum_{s=0}^{6} b_s p^s (1-p)^{6-s}.
\end{aligned}
$$

# Problem 1f

(f) Show that:

$$\sum_{s=0}^{6} b_s \ = \ 23.$$

**SOLUTION:** By inserting $p = \frac{1}{2}$ into the expression for $h(p)$ we get:

$$\begin{aligned} h(\frac{1}{2}) &= \sum_{s=0}^{6} b_s(\frac{1}{2})^s(1-(\frac{1}{2}))^{6-s} \\ &= \sum_{s=0}^{6} b_s(\frac{1}{2})^6 = [\sum_{s=0}^{6} b_s](\frac{1}{2})^6 = 23 \cdot (\frac{1}{2})^6, \end{aligned}$$

where the last equality follows by the last result in (c). Hence:

$$\sum_{s=0}^{6} b_s = 23.$$

# Problem 1g

(g) Determine $b_0, b_1, \ldots, b_6$.

**SOLUTION:** Since the smallest path sets have 3 components, we must have:

$$b_0 = b_1 = b_2 = 0.$$

We know from (a) that there are 4 minimal paths, all of size 3. Hence we have:

$$b_3 = 4.$$

Since all cut sets have at least 2 components, all sets of size 5 or 6 must be path sets. Hence, we have:

$$b_5 = \binom{6}{5} = 6, \quad \text{and} \quad b_6 = \binom{6}{6} = 1.$$

# Problem 1g (cont.)

In order to determine $b_4$, we could go through all sets of size 4, i.e., $\binom{6}{s} = 15$ sets, and count the path sets among these sets.

Alternatively, we can apply the result from (f). This gives us an equation which we can use to determine $b_4$:

$$0 + 0 + 0 + 4 + b_4 + 6 + 1 = 23$$

This implies that:

$$b_4 = 12.$$

**NOTE:** By the same arguments as we have used in this problem we can show more generally that if $(C, \phi)$ is a binary monotone system of order $n$, then:

$$\sum_{s=0}^{n} b_s = h(\frac{1}{2}) \cdot 2^n.$$

# Problem 2a

If $T_1, \ldots, T_n$ are random variables, and we let $\boldsymbol{T} = (T_1, \ldots, T_n)$, we say that $T_1, \ldots, T_n$ are *associated* if

$$\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0,$$

for all binary, non-decreasing functions $\Gamma$ and $\Delta$.

(a) Prove that non-decreasing functions of associated random variables are associated.

**SOLUTION:** Let $T_1, \ldots, T_n$ be associated, and let $\boldsymbol{T} = (T_1, \ldots, T_n)$. Moreover, we let:

$$S_i = f_i(\boldsymbol{T}), \quad i = 1, \ldots, m,$$

where $f_1, \ldots, f_m$ are non-decreasing functions. Let $\boldsymbol{S} = (S_1, \ldots, S_m)$.

# Problem 2a (cont.)

Finally, let:

$$\Gamma = \Gamma(\boldsymbol{S}) \quad \text{and} \quad \Delta = \Delta(\boldsymbol{S})$$

be binary non-decreasing functions.

Then:

$$\Gamma(\boldsymbol{S}) = \Gamma(f_1(\boldsymbol{T}), \ldots, f_m(\boldsymbol{T}))$$
$$\Delta(\boldsymbol{S}) = \Delta(f_1(\boldsymbol{T}), \ldots, f_m(\boldsymbol{T}))$$

are non-decreasing functions of $\boldsymbol{T}$ as well. Hence, by the definition of association, it follows that:

$$\text{Cov}(\Gamma(\boldsymbol{S}), \Delta(\boldsymbol{S})) = \text{Cov}(\Gamma(f_1(\boldsymbol{T}), \ldots, f_m(\boldsymbol{T})), \Delta(f_1(\boldsymbol{T}), \ldots, f_m(\boldsymbol{T}))) \geq 0.$$

Thus, we conclude that $S_1, \ldots, S_m$ are associated as well.

# Problem 2b

(b) Assume that $T_1, \ldots, T_n$ are associated random variables such that $0 \leq T_i \leq 1$, $i = 1, \ldots, n$.

Prove that:

$$E[\prod_{i=1}^{n} T_i] \geq \prod_{i=1}^{n} E[T_i]$$

$$E[\coprod_{i=1}^{n} T_i] \leq \coprod_{i=1}^{n} E[T_i].$$

**SOLUTION:**
We note that since $0 \leq T_i \leq 1$, both $T_i$ and $S_i = 1 - T_i$ are non-negative random variables, $i = 1, \ldots, n$.

Hence, the product functions $\prod_{i=1}^{n} T_i$ and $\prod_{i=1}^{n} S_i$ are both non-decreasing in each argument.

# Problem 2b (cont.)

Since non-decreasing functions of associated random variables have non-negative covariance, we find:

$$E[\prod_{i=1}^{n} T_i] - E[T_1]E[\prod_{i=2}^{n} T_i] = \mathrm{Cov}(T_1, \prod_{i=2}^{n} T_i) \geq 0,$$

since the product function is non-decreasing in each argument because $T_i \geq 0$, $i = 2, \ldots, n$. This implies that:

$$E[\prod_{i=1}^{n} T_i] \geq E[T_1]E[\prod_{i=2}^{n} T_i].$$

By repeated use of this argument we get:

$$E[\prod_{i=1}^{n} T_i] \geq E[T_1]E[\prod_{i=2}^{n} T_i] \geq E[T_1]E[T_2]E[\prod_{i=3}^{n} T_i] \geq \cdots \geq \prod_{i=1}^{n} E[T_i]$$

Thus, the first inequality is proved.

# Problem 2b (cont.)

It can be shown that non-increasing functions of associated random variables are associated. Thus, $S_1, \ldots, S_n$ are associated random variables.

Moreover, $0 \leq S_i \leq 1$, $i = 1, \ldots, n$.

Hence, we can apply the first inequality to these variables and get:

$$E[\coprod_{i=1}^{n} T_i] = 1 - E[\prod_{i=1}^{n}(1 - T_i)] = 1 - E[\prod_{i=1}^{n} S_i]$$

$$\leq 1 - \prod_{i=1}^{n} E(S_i) = 1 - \prod_{i=1}^{n}(1 - E[T_i]) = \coprod_{i=1}^{n} E[T_i]$$

Thus, the second inequality is proved as well.

# Problem 2c

(c) Interpret the inequalities in (b) by applying them to the binary component state variables $X_1, \ldots, X_n$.

**SOLUTION:**
Using the first inequality it follows that for a series system where the component state variables are associated, an incorrect assumption of independence implies that the system reliability is underestimated.

Using the second inequality it follows that for a parallel system where the component state variables are associated, an incorrect assumption of independence implies that the system reliability is overestimated.

# Problem 2d

(d) Let $X_1, \ldots, X_n$ be the associated component state variable of a binary monotone system $(C, \phi)$.

Furthermore, let $(P_1, \rho_1), \ldots, (P_p, \rho_p))$ be the minimal path series structures be, and let $(K_1, \kappa_1), \ldots, (K_k, \kappa_k))$ be the minimal cut parallel structures.

Prove that:

$$\prod_{j=1}^{k} P(\kappa_j(\boldsymbol{X}^{K_j}) = 1) \leq h \leq \coprod_{j=1}^{p} P(\rho_j(\boldsymbol{X}^{P_j}) = 1).$$

# Problem 2d (cont.)

**SOLUTION:** Since non-decreasing functions of associated random variables are associated from a), it follows that the minimal path series structures, and the minimal cut parallel structures, are associated.

Hence, we get:

$$\begin{aligned}\prod_{j=1}^{k} P(\kappa_j(\boldsymbol{X}^{K_j}) = 1) &= \prod_{j=1}^{k} E[\kappa_j(\boldsymbol{X}^{K_j})] \leq E[\prod_{j=1}^{k} \kappa_j(\boldsymbol{X}^{K_j})] = h \\ &= E[\prod_{j=1}^{p} \rho_j(\boldsymbol{X}^{P_j})] \leq \prod_{j=1}^{p} E[\rho_j(\boldsymbol{X}^{P_j}] = \prod_{j=1}^{p} P(\rho_j(\boldsymbol{X}^{P_j}) = 1),\end{aligned}$$

where the first inequality follows from the first inequality in (b), the first and second equalities follow from the representation of the system via its minimal path series and cut parallel structures and the final inequality follows from the second inequality in (b).

# Problem 2e

(e) Make the same assumptions as in (d), and assume in addition that the component state variables are independent with component reliabilities $p_1, p_2, \dots p_n$. Use the result in (d) to prove that:

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\boldsymbol{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i.$$

**SOLUTION:**
If $X_1, \dots, X_n$ are independent, we have:

$$P(\kappa_j(\boldsymbol{X}^{K_j}) = 1) = E[\coprod_{i \in K_j} X_i] = \coprod_{i \in K_j} p_i, \quad j = 1, \dots, k.$$

$$P(\rho_j(\boldsymbol{X}^{P_j}) = 1) = E[\rho_j(\boldsymbol{X}^{P_j}) = 1)] = \prod_{i \in P_j} p_i, \quad j = 1, \dots, p.$$

The result follows by inserting these expressions into the inequalities in (d).

# Problem 2f

(f) Consider the system in Problem 1. Assume that all components have the same component reliability $p = 0.9$.

Compute the bounds derived in (e) and comment on how well they approximate the actual system reliability in this case.

**SOLUTION:** We start out by recalling the expression for $h(p)$ derived in 1(c):

$$h(p) = p^4 \cdot (2 - p)^2 + 2p^3(1 - p)(2 - p^2)$$

# Problem 2f (cont.)

Considering the minimal cut sets from 1(a), the 3 first sets are of size 2, while the 4 last sets of size 3. Hence, we have:

$$\coprod_{i \in K_1} p = \coprod_{i \in K_2} p = \coprod_{i \in K_3} p = p \amalg p = (2p - p^2)$$

$$\coprod_{i \in K_4} p = \coprod_{i \in K_5} p = \coprod_{i \in K_6} p = \coprod_{i \in K_7} p = p \amalg p \amalg p = (3p - 3p^2 + p^3)$$

Considering the minimal path sets from 1(a), all 4 sets are of size 3. Hence, we have:

$$\prod_{i \in P_1} p = \cdots = \prod_{i \in P_4} p = p^3$$

# Problem 2f (cont.)

Hence, the lower and upper bounds become:

$$\ell(p) = \prod_{j=1}^{7} \coprod_{i \in K_j} p = (2p - p^2)^3 \cdot (3p - 3p^2 + p^3)^4$$

$$u(p) = \coprod_{j=1}^{4} \prod_{i \in P_j} p = \coprod_{j=1}^{4} p^3 = 1 - (1 - p^3)^4$$

By inserting $p = 0.9$ we get:

$$\ell(0.9) = 0.9664,$$
$$h(0.9) = 0.9674,$$
$$u(0.9) = 0.9946.$$

We observe that the lower bound is very close to the correct system reliability, while the upper bound is noticeably higher.

# Problem 2g

(g) In the previous points we have derived upper and lower bounds for the system reliability. In which cases is it important to have such bounds?

**SOLUTION:** In real-life cases the exact system reliability may not be possible to calculate.

This may be the case for large, complex systems where the computations simply take too much time, but also for systems where the component state varaibles are dependent.

In such cases bounds for the system reliability are important.