Exam STK3405/4405 - 2021

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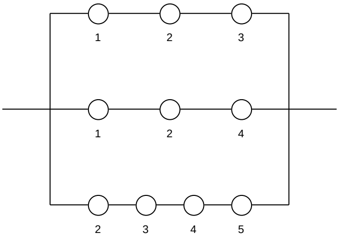
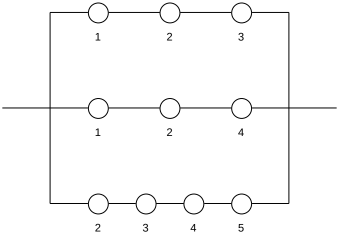
Problem 1



Binary monotone system (C, ϕ) with component set of the system is $C = \{1, \ldots, 5\}$. $\boldsymbol{X} = (X_1, \ldots, X_5)$ is the vector of component state variables, where $X_1, \ldots, X_5$ are stochastically independent.

Let $\boldsymbol{p} = (p_1, \ldots, p_5)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, \ldots, 5$.

Problem 1a. Minimal path and cut sets of the system



(a) Find the minimal path sets (3 sets) and the minimal cut sets (5 sets) of the system.

SOLUTION:

Minimal path sets: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{2, 3, 4, 5\}$

Minimal cut sets: $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2\}$, $\{3, 4\}$ ■

Problem 1b

(b) We let $h(\boldsymbol{p}) = P(\phi = 1)$ denote the reliability function of the system. Show that:

$$h(\boldsymbol{p}) = p_2 \cdot [p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5]$$

SOLUTION: We note that component 2 is in series with the rest of the system. Hence, $h(0_2, \boldsymbol{p}) = 0$, and we get:

$$h(\boldsymbol{p}) = p_2 \cdot h(1_2, \boldsymbol{p}) + (1 - p_2) h(0_2, \boldsymbol{p}) = p_2 \cdot h(1_2, \boldsymbol{p})$$

In order to find $h(1_2, \boldsymbol{p})$ we do a pivotal decomposition with respect to component 1. If component 1 is functioning, the rest of the system is a parallel connection of components 3 and 4, while if 1 is failed, the rest of the system is a series connection of components 3, 4 and 5. Hence, we get:

$$\begin{aligned} h(\boldsymbol{p}) &= p_2 \cdot h(1_2, \boldsymbol{p}) \\ &= p_2 \cdot [p_1 \cdot h(1_1, 1_2, \boldsymbol{p}) + (1 - p_1) \cdot h(0_1, 1_2, \boldsymbol{p})] \\ &= p_2 \cdot [p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5] \qquad \blacksquare \end{aligned}$$

Problem 1c

The Birnbaum measure for the *reliability importance* of component i is defined as:

$$I_B^{(i)} = P(\text{Component } i \text{ is critical for the system}), \quad i = 1, 2, \ldots, 5.$$

Show that:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \quad i = 1, 2, \ldots, 5.$$

SOLUTION: Component i is *critical* for the system if and only if:

$$\phi(1_i, \boldsymbol{X}) = 1, \text{ and } \phi(0_i, \boldsymbol{X}) = 0 \tag{1}$$

Since ϕ is non-decreasing in each argument, we always have that: $\phi(1_i, \boldsymbol{X}) \geq \phi(0_i, \boldsymbol{X})$. Thus, the condition (1) is equivalent to:

$$\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1 \tag{2}$$

Problem 1c (cont.)

Hence, since the component state variables are assumed to be independent, it follows for $i = 1, 2, \ldots, 5$ that:

$$
\begin{aligned}
I_B^{(i)} &= P(\text{Component } i \text{ is critical for the system}) \\
&= P(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1) \\
&= E[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})] \\
&= E[\phi(1_i, \boldsymbol{X})] - E[\phi(0_i, \boldsymbol{X})] \\
&= h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p}) \\
&= \frac{\partial}{\partial p_i}[p_i \cdot h(1_i, \boldsymbol{p}) + (1 - p_i) \cdot h(0_i, \boldsymbol{p})] \\
&= \frac{\partial h(\boldsymbol{p})}{\partial p_i} \qquad \blacksquare
\end{aligned}
$$

Problem 1d

(d) Show that:

$$I_B^{(2)} = p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5$$

$$I_B^{(5)} = (1 - p_1) \cdot p_2 p_3 p_4$$

SOLUTION: By using the result from (c) we get:

$$I_B^{(2)} = \frac{\partial}{\partial p_2} [p_2 \cdot [p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5]]$$

$$= p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5$$

$$I_B^{(5)} = \frac{\partial}{\partial p_5} [p_2 \cdot [p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5]]$$

$$= (1 - p_1) \cdot p_2 p_3 p_4 \qquad \blacksquare$$

Problem 1e

In the remaining part of this problem we assume that $0 < p_i < 1$, $i = 1, 2, \dots, 5$.

(e) Show that if $p_5 \geq p_2$, then $I_B^{(2)} > I_B^{(5)}$.

SOLUTION: In order to compare $I_B^{(2)}$ and $I_B^{(5)}$, we consider:

$$
\begin{aligned}
I_B^{(2)} - I_B^{(5)} &= p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5 - (1 - p_1) \cdot p_2 p_3 p_4 \\
&= p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 (p_5 - p_2)
\end{aligned}
$$

If $p_5 \geq p_2$, we observe that both terms in the difference between $I_B^{(2)}$ and $I_B^{(5)}$ are non-negative. Moreover, since we have assumed that $0 < p_i < 1$ for all i. the first term is strictly positive. Hence, we conclude $I_B^{(2)} > I_B^{(5)}$ ■

Problem 1f

(f) Show that if $p_1 \geq \frac{1}{2}$, then $I_B^{(2)} > I_B^{(5)}$.

SOLUTION: In order to compare $I_B^{(2)}$ and $I_B^{(5)}$, we again consider:

$$I_B^{(2)} - I_B^{(5)} = p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 (p_5 - p_2)$$

Since we have assumed that $0 < p_i < 1$ for all i, it follows that $(1 - p_1) \cdot p_3 p_4 > 0$ and that $(p_5 - p_2) > -1$. Hence, when $p_1 \geq \frac{1}{2}$, we get that:

$$\begin{aligned} I_B^{(2)} - I_B^{(5)} &> p_1 \cdot (p_3 \amalg p_4) - (1 - p_1) \cdot p_3 p_4 \\ &\geq \frac{1}{2} \cdot (p_3 \amalg p_4) - \frac{1}{2} \cdot p_3 p_4 \\ &= \frac{1}{2}(p_3 + p_4 - p_3 p_4 - p_3 p_4) \\ &= \frac{1}{2}[p_3(1 - p_4) + p_4(1 - p_3)] > 0 \end{aligned}$$

where the last inequality again follows since $0 < p_i < 1$ for all i. From this we conclude that $I_B^{(2)} > I_B^{(5)}$ ■

Problem 1g

(g) In this point we assume more specifically that $p_1 = p_5 = \frac{1}{10}$ and that $p_2 = p_3 = p_4 = \frac{9}{10}$. Calculate $I_B^{(2)}$ and $I_B^{(5)}$ and compare the results. Comment your findings.

SOLUTION: By using the result from (d) we get:

$$
\begin{aligned}
I_B^{(2)} &= p_1 \cdot (p_3 \amalg p_4) + (1 - p_1) \cdot p_3 p_4 p_5 \\
&= \frac{1}{10} \cdot (\frac{9}{10} + \frac{9}{10} - \frac{9}{10}\frac{9}{10}) + \frac{9}{10} \cdot \frac{9}{10}\frac{9}{10}\frac{1}{10} \\
&= \frac{990}{10000} + \frac{729}{10000} = \frac{1719}{10000}
\end{aligned}
$$

$$
\begin{aligned}
I_B^{(5)} &= (1 - p_1) \cdot p_2 p_3 p_4 \\
&= \frac{9}{10} \cdot \frac{9}{10}\frac{9}{10}\frac{9}{10} = \frac{6561}{10000}
\end{aligned}
$$

Problem 1g (cont.)

Thus, in this case we have $I_B^{(2)} < I_B^{(5)}$ which is the opposite ranking compared to the cases considered in the two previous points. We note that with these component reliabilities we have:

$p_1 < \frac{1}{2}$ (Thus, the result from (f) does not apply in this case)

$p_5 < p_2$ (Thus, the result from (e) does not apply in this case)

Indeed when p_1 is small, then the system is, with a high probability, reduced to a *series connection* of the components $2, 3, 4, 5$. In a series system the most important component is the one with the smallest reliability, i.e., component 5 in this case ■

Problem 1h

The Birnbaum measure for the *structural importance* of component i is defined as:

$$J_B^{(i)} = \frac{1}{2^{5-1}} \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})], \quad i = 1, 2, \ldots, 5.$$

(h) Explain briefly why: $J_B^{(2)} > J_B^{(i)}$ for $i = 1, 3, 4, 5$.

SOLUTION: We observe that component 2 is in series with the rest of the system, while none of the other components are in series with the rest of the system.

From this it follows that the structural importance of component 2 is greater than the structural importance of any of the other components. See Exercise 5.3 in the textbook.

On the next slide we include a formal proof of this result.

Problem 1h (cont.)

Since component 2 is in series with the rest of the system, we have:

$$\phi(0_2, \boldsymbol{x}) = 0, \qquad \text{for all } (\cdot_2, \boldsymbol{x}) \in \{0,1\}^4$$

We then choose another component $j \neq 2$. Since j is *not* in series with the rest of the system, we have:

$$\phi(0_j, \boldsymbol{x}) = 1, \qquad \text{for at least one } (\cdot_j, \boldsymbol{x}) \in \{0,1\}^4$$

Hence, we then get:

$$2^4 J_B^{(2)} = \sum_{(\cdot_2, \boldsymbol{X})} [\phi(1_2, \boldsymbol{x}) - \phi(0_2, \boldsymbol{x})] = \sum_{(\cdot_2, \boldsymbol{X})} [\phi(1_2, \boldsymbol{x}) + \phi(0_2, \boldsymbol{x})]$$

$$= \sum_{\boldsymbol{X}} \phi(\boldsymbol{x}) = \sum_{(\cdot_j, \boldsymbol{X})} [\phi(1_j, \boldsymbol{x}) + \phi(0_j, \boldsymbol{x})]$$

$$> \sum_{(\cdot_j, \boldsymbol{X})} [\phi(1_j, \boldsymbol{x}) - \phi(0_j, \boldsymbol{x})] = 2^4 J_B^{(j)} \qquad \blacksquare$$

Problem 2a

Let $X_1, \ldots, X_n$ be n binary associated random variables.

(a) Show that:

$$E[\prod_{i=1}^{n} X_i] \geq \prod_{i=1}^{n} E[X_i] \tag{3}$$

$$E[\coprod_{i=1}^{n} X_i] \leq \coprod_{i=1}^{n} E[X_i] \tag{4}$$

SOLUTION: We introduce $\boldsymbol{X} = (X_1, \ldots, X_n)$. Since $X_1, \ldots, X_n$ are binary associated random variables, we know that:

$$\text{Cov}(\Gamma(\boldsymbol{X}), \Delta(\boldsymbol{X})) \geq 0,$$

for all binary, non-decreasing functions, Γ and Δ.

Problem 2a(cont.)

Hence, in particular:

$$\mathrm{Cov}(X_1, \prod_{i=2}^{n} X_i) = E[\prod_{i=1}^{n} X_i] - E[X_1] \cdot E[\prod_{i=2}^{n} X_i] \geq 0$$

Thus, it follows that:

$$E[\prod_{i=1}^{n} X_i] \geq E[X_1] \cdot E[\prod_{i=2}^{n} X_i]$$

Repeated use of the same argument yields that:

$$E[\prod_{i=1}^{n} X_i] \geq E[X_1] \cdot E[\prod_{i=2}^{n} X_i] \geq E[X_1] \cdot E[X_2] \cdot E[\prod_{i=3}^{n} X_i] \geq \cdots \geq \prod_{i=1}^{n} E[X_i],$$

and thus, (3) is proved.

Problem 2a(cont.)

In order to prove (4) we note that since $X_1, \ldots, X_n$ are binary associated random variables, it follows that $(1 - X_1), \ldots, (1 - X_n)$ are binary associated random variables as well. Hence, by using (3) it follows that:

$$E[\prod_{i=1}^{n}(1 - X_i)] \ \geq \ \prod_{i=1}^{n}(1 - E[X_i]).$$

Hence, we get that:

$$E[\coprod_{i=1}^{n} X_i] = 1 - E[\prod_{i=1}^{n}(1 - X_i)] \leq 1 - \prod_{i=1}^{n}(1 - E[X_i]) = \coprod_{i=1}^{n} E[X_i],$$

and thus, (4) is proved as well ■

Problem 2b

Let $X_1, \ldots, X_n$ be the associated component state variables of a binary monotone system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$.

(b) Show that:

$$\max_{1 \leq j \leq p} \prod_{i \in P_j} E[X_i] \;\leq\; E[\phi] \;\leq\; \min_{1 \leq j \leq k} \coprod_{i \in K_j} E[X_i] \tag{5}$$

SOLUTION: We have that:

$$\min_{i \in P_r} X_i \leq \max_{1 \leq r \leq p} \min_{i \in P_r} X_i = \phi(\boldsymbol{X}) = \min_{1 \leq s \leq k} \max_{i \in K_s} X_i \leq \max_{i \in K_s} X_i,$$

for all $r = 1, \ldots, p$ and all $s = 1, \ldots, k$. This implies that:

$$E[\min_{i \in P_r} X_i] \;\leq\; E[\phi] \;\leq\; E[\max_{i \in K_s} X_i]$$

for all $r = 1, \ldots, p$ and all $s = 1, \ldots, k$.

Problem 2b (cont.)

Hence, we must have:

$$\max_{1 \leq j \leq p} E[\min_{i \in P_r} X_i] \ \leq \ E[\phi] \ \leq \ \min_{1 \leq j \leq k} E[\max_{i \in K_s} X_i].$$

Furthermore, since $X_1, \ldots, X_n$ are associated, we may use the result from (a) and get:

$$E[\min_{i \in P_r} X_i] = E[\prod_{i \in P_r} X_i] \geq \prod_{i \in P_r} E[X_i]$$

$$E[\max_{i \in K_s} X_i] = E[\coprod_{i \in K_s} X_i] \leq \coprod_{i \in K_s} E[X_i]$$

Inserting these inequalities into the bounds for $E[\phi]$ we get:

$$\max_{1 \leq j \leq p} \prod_{i \in P_j} E[X_i] \ \leq \ E[\phi] \ \leq \ \min_{1 \leq j \leq k} \coprod_{i \in K_j} E[X_i]$$

and thus, (5) is proved ■

Problem 2c

(c) Show that:

$$\prod_{j=1}^{k} E[\coprod_{i \in K_j} X_i] \;\leq\; E[\phi] \;\leq\; \coprod_{j=1}^{p} E[\prod_{i \in P_j} X_i]. \tag{6}$$

SOLUTION: We introduce:

$$\rho_j(\boldsymbol{X}) = \prod_{i \in P_j} X_i, \quad j = 1, \ldots, p,$$

$$\kappa_j(\boldsymbol{X}) = \coprod_{i \in K_j} X_i, \quad j = 1, \ldots, k.$$

Since $\rho_1, \ldots, \rho_p$ and $\kappa_1, \ldots, \kappa_k$ are non-decreasing functions of $\boldsymbol{X}$, they are associated.

Problem 2c (cont.)

Hence, by the result in (a) we have:

$$E[\phi] = E[\coprod_{j=1}^{p} \prod_{i \in P_j} X_i] = E[\coprod_{j=1}^{p} \rho_j(\boldsymbol{X})] \leq \coprod_{j=1}^{p} E[\rho_j(\boldsymbol{X})] = \coprod_{j=1}^{p} E[\prod_{i \in P_j} X_i]$$

$$E[\phi] = E[\prod_{j=1}^{k} \coprod_{i \in K_j} X_i] = E[\prod_{j=1}^{k} \kappa_j(\boldsymbol{X})] \geq \prod_{j=1}^{k} E[\kappa_j(\boldsymbol{X})] = \prod_{j=1}^{k} E[\coprod_{i \in K_j} X_i]$$

Hence, (6) is proved ■

- The lower and upper bounds on $E[\phi]$ given in (5) are denoted L_1 and U_1 respectively.
- The lower and upper bounds on $E[\phi]$ given in (6) are denoted L_2 and U_2 respectively.

Problem 2d

In the rest of this problem we assume that (C, ϕ) is a 2-out-of-3 system. That is, $C = \{1, 2, 3\}$ and the structure function, ϕ, is given by:

$$\phi(\boldsymbol{X}) = X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3,$$

where $\boldsymbol{X} = (X_1, X_2, X_3)$. Moreover, we assume that the joint distribution of the component state variables satisfies the following properties:

$$E[X_1] = E[X_2] = E[X_3] = p,$$

$$E[X_1X_2] = E[X_1X_3] = E[X_2X_3] = p^{2-\alpha},$$

$$E[X_1X_2X_3] = p^{3-2\alpha},$$

where $0 < p < 1$ and $0 \leq \alpha \leq 1$. It can be shown that these properties imply that X_1, X_2, X_3 are *associated random variables*.

Problem 2d (cont.)

(d) We now consider the correlation between the component state variables. Show that:

$$\mathrm{Corr}(X_i, X_j) = \frac{p^{2-\alpha} - p^2}{p(1-p)}, \quad \text{for } i \neq j.$$

Moreover, show that the correlation is increasing in α. In particular, calculate the correlation for the cases $\alpha = 0$ and $\alpha = 1$. Comment your findings.

SOLUTION: For $i \neq j$ we have that:

$$\mathrm{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = E[X_i] - (E[X_i])^2 = p - p^2 = p(1-p)$$

$$\mathrm{Var}(X_j) = E[X_j^2] - (E[X_j])^2 = E[X_j] - (E[X_j])^2 = p - p^2 = p(1-p)$$

$$\mathrm{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = p^{2-\alpha} - p^2$$

Hence, we get that:

$$\mathrm{Corr}(X_i, X_j) = \frac{\mathrm{Cov}(X_i, X_j)}{\sqrt{\mathrm{Var}(X_i) \cdot \mathrm{Var}(X_j)}} = \frac{p^{2-\alpha} - p^2}{p(1-p)}$$

Problem 2d (cont.)

(d) We now consider the correlation between the component state variables. Show that:

$$\text{Corr}(X_i, X_j) = \frac{p^{2-\alpha} - p^2}{p(1-p)}, \quad \text{for } i \neq j.$$

Moreover, show that the correlation is increasing in α. In particular, calculate the correlation for the cases $\alpha = 0$ and $\alpha = 1$. Comment your findings.

SOLUTION: For $i \neq j$ we have that:

$$\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = E[X_i] - (E[X_i])^2 = p - p^2 = p(1-p)$$

$$\text{Var}(X_j) = E[X_j^2] - (E[X_j])^2 = E[X_j] - (E[X_j])^2 = p - p^2 = p(1-p)$$

$$\text{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = p^{2-\alpha} - p^2$$

Hence, we get that:

$$\text{Corr}(X_i, X_j) = \frac{\text{Cov}(X_i, X_j)}{\sqrt{\text{Var}(X_i) \cdot \text{Var}(X_j)}} = \frac{p^{2-\alpha} - p^2}{p(1-p)}$$

Problem 2d (cont.)

Showing that $\mathrm{Corr}(X_i, X_j)$ is increasing in α is equivalent to showing that $p^{2-\alpha}$ is increasing in α.

In order to show this, we compute the derivative of $p^{2-\alpha}$ with respect to α:

$$\frac{\partial}{\partial\alpha} p^{2-\alpha} = (-\ln(p))p^{2-\alpha}$$

Since $0 < p < 1$, we have that $(-\ln(p)) > 0$.

Thus, the derivative of $p^{2-\alpha}$ with respect to α is positive, which implies that $p^{2-\alpha}$ is increasing in α.

Problem 2d (cont.)

If $\alpha = 0$ we get:

$$\mathrm{Corr}(X_i, X_j) = \frac{p^{2-\alpha} - p^2}{p(1-p)} = \frac{p^2 - p^2}{p(1-p)} = 0$$

Thus, in this case X_i and X_j are independent.

If $\alpha = 1$ we get:

$$\mathrm{Corr}(X_i, X_j) = \frac{p^{2-\alpha} - p^2}{p(1-p)} = \frac{p(1-p)}{p(1-p)} = 1$$

Thus, in this case X_i and X_j are completely dependent ■

Problem 2e

(e) Show that:

$$L_1 = p^2 \quad \text{and} \quad U_1 = 1 - (1-p)^2$$

and that:

$$L_2 = (2p - p^{2-\alpha})^3 \quad \text{and} \quad U_2 = 1 - (1 - p^{2-\alpha})^3$$

and that:

$$E[\phi] = 3p^{2-\alpha} - 2p^{3-2\alpha}$$

SOLUTION: Since (C, ϕ) is a 2-out-of-3 system we have:

- Minimal path sets: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$
- Minimal cut sets: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$

Problem 2e (cont.)

Hence, by using the properties of the joint distribution of X_1, X_2 and X_3 we get:

$$L_1 = \max_{1 \le j \le 3} \prod_{i \in P_j} E[X_i] = p^2$$

$$U_1 = \min_{1 \le j \le 3} \coprod_{i \in K_j} E[X_i] = 1 - (1-p)^2$$

Furthermore,

$$L_2 = \prod_{j=1}^{3} E[\coprod_{i \in K_j} X_i] = \prod_{j=1}^{3} E[X_1 + X_2 - X_1 X_2] = (2p - p^{2-\alpha})^3$$

$$U_2 = \coprod_{j=1}^{3} E[\prod_{i \in P_j} X_i] = \coprod_{j=1}^{3} E[X_1 X_2] = 1 - (1 - p^{2-\alpha})^3$$

Finally,

$$E[\phi] = E[X_1 X_2 + X_1 X_3 + X_2 X_3 - 2X_1 X_2 X_3] = 3p^{2-\alpha} - 2p^{3-2\alpha}$$

Problem 2e (cont.)

Hence, by using the properties of the joint distribution of X_1, X_2 and X_3 we get:

$$L_1 = \max_{1 \le j \le 3} \prod_{i \in P_j} E[X_i] = p^2$$

$$U_1 = \min_{1 \le j \le 3} \coprod_{i \in K_j} E[X_i] = 1 - (1-p)^2$$

Furthermore,

$$L_2 = \prod_{j=1}^{3} E[\coprod_{i \in K_j} X_i] = \prod_{j=1}^{3} E[X_1 + X_2 - X_1 X_2] = (2p - p^{2-\alpha})^3$$

$$U_2 = \coprod_{j=1}^{3} E[\prod_{i \in P_j} X_i] = \coprod_{j=1}^{3} E[X_1 X_2] = 1 - (1 - p^{2-\alpha})^3$$

Finally,

$$E[\phi] = E[X_1 X_2 + X_1 X_3 + X_2 X_3 - 2X_1 X_2 X_3] = 3p^{2-\alpha} - 2p^{3-2\alpha} \quad \blacksquare$$

Problem 2e (cont.)

Hence, by using the properties of the joint distribution of X_1, X_2 and X_3 we get:

$$L_1 = \max_{1 \leq j \leq 3} \prod_{i \in P_j} E[X_i] = p^2$$

$$U_1 = \min_{1 \leq j \leq 3} \coprod_{i \in K_j} E[X_i] = 1 - (1-p)^2$$

Furthermore,

$$L_2 = \prod_{j=1}^{3} E[\coprod_{i \in K_j} X_i] = \prod_{j=1}^{3} E[X_1 + X_2 - X_1 X_2] = (2p - p^{2-\alpha})^3$$

$$U_2 = \coprod_{j=1}^{3} E[\prod_{i \in P_j} X_i] = \coprod_{j=1}^{3} E[X_1 X_2] = 1 - (1 - p^{2-\alpha})^3$$

Finally,

$$E[\phi] = E[X_1 X_2 + X_1 X_3 + X_2 X_3 - 2 X_1 X_2 X_3] = 3p^{2-\alpha} - 2p^{3-2\alpha} \qquad \blacksquare$$

Problem 2f

(f) Show that L_2 is decreasing in α while U_2 is increasing in α. What can you say about the quality of these bounds when the correlation between the component state variables increases?

SOLUTION: We recall from (e) that:

$$L_2 = (2p - p^{2-\alpha})^3$$

$$U_2 = 1 - (1 - p^{2-\alpha})^3$$

We observe that L_2 is *decreasing* in α is equivalent to that $p^{2-\alpha}$ is *increasing* in α which was shown in (d). Similarly, that U_2 is *increasing* in α is equivalent to that $p^{2-\alpha}$ is increasing in α, which was also shown in (d).

When the lower bound, L_2, is decreasing, while the upper bound U_2, is increasing, the difference between L_2 and U_2 is increasing. Thus, the quality of these bounds become worse when the correlation between the component state variables increases ■

Problem 2f

(f) Show that L_2 is decreasing in α while U_2 is increasing in α. What can you say about the quality of these bounds when the correlation between the component state variables increases?

SOLUTION: We recall from (e) that:

$$L_2 = (2p - p^{2-\alpha})^3$$

$$U_2 = 1 - (1 - p^{2-\alpha})^3$$

We observe that L_2 is *decreasing* in α is equivalent to that $p^{2-\alpha}$ is *increasing* in α which was shown in (d). Similarly, that U_2 is *increasing* in α is equivalent to that $p^{2-\alpha}$ is increasing in α, which was also shown in (d).

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Problem 2g

(g) Assume that $\alpha = 1$. Show that we in this case have:

$$L_2 \; < \; L_1 \; < \; E[\phi] \; < \; U_1 \; < \; U_2$$

Which bounds would you recommend in this case?

SOLUTION: When $\alpha = 1$, we get that:

$$L_2 = (2p - p^{2-\alpha})^3 = (2p - p)^3 = p^3$$

$$U_2 = 1 - (1 - p^{2-\alpha})^3 = 1 - (1 - p)^3$$

$$E[\phi] = 3p^{2-\alpha} - 2p^{3-2\alpha} = 3p - 2p = p$$

At the same time $L_1 = p^2$ while $U_1 = 1 - (1 - p)^2$ (since these bounds do not depend on α).

Problem 2g (cont.)

Combining all this, and the assumption that $0 < p < 1$, we get:

$$p^3 < p^2 < p < 1 - (1-p)^2 < 1 - (1-p)^3$$

Hence, it follows that:

$$L_2 \; < \; L_1 \; < \; E[\phi] \; < \; U_1 \; < \; U_2$$

Obviously the bounds should to be chosen as close as possible to the true value, $E[\phi]$. Thus, we recommend using L_1 as lower bound and U_1 as upper bound ■

Problem 2h

(h) Assume that $\alpha = 0$. What kind of bounds would you recommend in this case?

SOLUTION: When $\alpha = 0$, we get that:

$$L_2 = (2p - p^{2-\alpha})^3 = (2p - p^2)^3$$

$$U_2 = 1 - (1 - p^{2-\alpha})^3 = 1 - (1 - p^2)^3$$

$$E[\phi] = 3p^{2-\alpha} - 2p^{3-2\alpha} = 3p^2 - 2p^3$$

At the same time $L_1 = p^2$ while $U_1 = 1 - (1 - p)^2$ (since these bounds do not depend on α).

Problem 2h (cont.)

In this case it can be shown that $L_1 < L_2$ for some values of p while the opposite inequality holds for other values of p.

Similarly, it can be shown that $U_1 < U_2$ for some values of p while the opposite inequality holds for other values of p.

To ensure that we get the best bounds, we recommend using the lower bound L^* and the upper bound U^* given by:

$$L^* = \max(L_1, L_2)$$

$$U^* = \min(U_1, U_2) \qquad \blacksquare$$