# STK3405 – Exercises Chapter 2

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# Exercise 2.1

What is the reliability function of a series system of order $n$ where the component states are assumed to be independent?

SOLUTION: A series system of order $n$ is a binary monotone system $(C, \phi)$ which functions if and only if all the $n$ components function. Thus, the structure function of the system is:

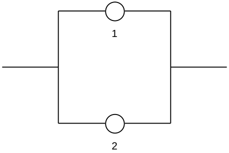
$$\phi(\boldsymbol{X}) = \prod_{i=1}^{n} X_i.$$

Assuming that the component state variables $X_1, \ldots, X_n$ are independent with reliabilites $p_1, \ldots, p_n$ respectively, we get that:

$$h(\boldsymbol{p}) = P(\phi(\boldsymbol{X}) = 1) = E[\phi(\boldsymbol{X})] = E[\prod_{i=1}^{n} X_i]$$

$$= \prod_{i=1}^{n} E[X_i] \text{ (using the independence) } = \prod_{i=1}^{n} p_i$$

# Exercise 2.2




Consider the parallel structure of order 2. Assume that the component states are independent. The reliability function of this system is:

$$h(\boldsymbol{p}) = p_1 \amalg p_2 = 1 - (1 - p_1)(1 - p_2) = p_1 + p_2 - p_1 p_2.$$

a) If you know that $p_1 = P(X_1 = 1) = 0.5$ and $p_2 = P(X_2 = 1) = 0.7$, what is the reliability of the parallel system?

SOLUTION: $h(0.5, 0.7) = 0.5 + 0.7 - 0.5 \cdot 0.7 = 0.85$

# Exercise 2.2 (cont.)

b) What is the system reliability if $p_1 = 0.9$ and $p_2 = 0.1$?

SOLUTION: $h(0.9, 0.1) = 0.9 + 0.1 - 0.9 \cdot 0.1 = 0.91$

c) Can you give an interpretation of these results?

SOLUTION: For a parallel system it is better to have *one really good* component and *one really bad* component, than to have two components with reliabilities close 0.5.

# Exercise 2.2 (cont.)

Assume more generally that $P(X_1 = 1) = p$ and $P(X_2 = 1) = 1 - p$. This implies that:

$$h(\boldsymbol{p}) = p + (1 - p) - p(1 - p) = 1 - p(1 - p).$$

This is parabola with minimum at $p = 0.5$. Thus, the system reliability of the parallel system is smallest when the components have the same reliability.

For a series system of two components with $P(X_1 = 1) = p$ and $P(X_2 = 1) = 1 - p$ we have:

$$h(\boldsymbol{p}) = p(1 - p).$$

This is parabola with maximum at $p = 0.5$. Thus, the system reliability of the series system is largest when the components have the same reliability.

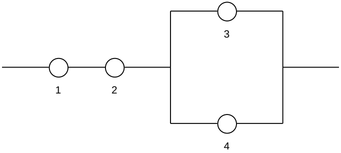
# Exercise 2.3

Consider a binary monotone system $(C, \phi)$, where the component set is $C = \{1, \ldots, 4\}$ and where $\phi$ is given by:

$$\phi(\boldsymbol{x}) = x_1 \cdot x_2 \cdot (x_3 \amalg x_4).$$

a) Draw a reliability block diagram of this system.

SOLUTION:



# Exercise 2.3 (cont.)

b) Assume again that $\phi$ is given by:

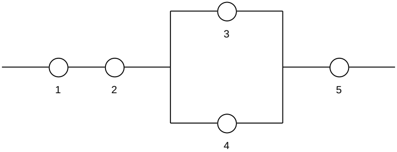
$$\phi(\boldsymbol{x}) = x_1 \cdot x_2 \cdot (x_3 \amalg x_4).$$

and that $X_1, X_2, X_3, X_4$ are independent with reliabilities $p_1, p_2, p_3, p_4$ respectively. What is the corresponding reliability function?

SOLUTION:

$$\begin{aligned} h(\boldsymbol{p}) &= E[\phi(\boldsymbol{X})] = E[X_1 \cdot X_2 \cdot (X_3 \amalg X_4)] \\ &= E[X_1] \cdot E[X_2] \cdot (E[X_3] \amalg E[X_4]) \quad \text{using independence} \\ &= p_1 \cdot p_2 \cdot (p_3 \amalg p_4) \end{aligned}$$

# Exercise 2.4




a) What is the structure function of this system?

SOLUTION: The structure function of this system is:

$$\phi(\boldsymbol{x}) = x_1 \cdot x_2 \cdot (x_3 \amalg x_4) \cdot x_5.$$

# Exercise 2.4 (cont.)

b) Assume again that $\phi$ is given by:

$$\phi(\boldsymbol{x}) = x_1 \cdot x_2 \cdot (x_3 \amalg x_4) \cdot x_5.$$

and that $X_1, X_2, X_3, X_4, X_5$ are independent with reliabilities $p_1, p_2, p_3, p_4, p_5$ respectively. What is the corresponding reliability function?

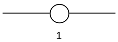
SOLUTION:

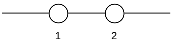
$$\begin{aligned} h(\boldsymbol{p}) &= E[\phi(\boldsymbol{X})] = E[X_1 \cdot X_2 \cdot (X_3 \amalg X_4) \cdot X_5] \\ &= E[X_1] \cdot E[X_2] \cdot (E[X_3] \amalg E[X_4]) \cdot E[X_5] \quad \text{using independence} \\ &= p_1 \cdot p_2 \cdot (p_3 \amalg p_4) \cdot p_5 \end{aligned}$$

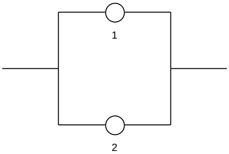
# Exercise 2.5

There are 8 different coherent systems of order less than or equal to 3 (not counting permutations in the numbering of components). What are they?

SOLUTION: There are three coherent systems of order 1 or 2:

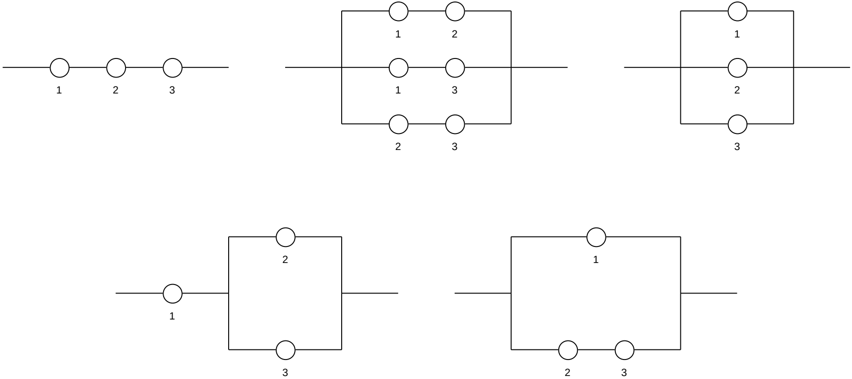


# Exercise 2.5 (cont.)

There are five coherent systems of order 3:



# Exercise 2.6

Consider a monotone system $(C, \phi)$ of order $n$, and let $A \subset C$ be the set of irrelevant components.

Furthermore, let $(C \setminus A, \phi')$, be a binary monotone system of order $m = n - |A|$, where $\phi'$ is defined for all $m$-dimensional binary vectors $\boldsymbol{x} \in \{0, 1\}^m$ as:

$$\phi'(\boldsymbol{x}) = \phi(\mathbf{1}^A, \boldsymbol{x}^{C \setminus A})$$

Show that $(C \setminus A, \phi')$ is coherent.

SOLUTION: Since $\phi$ is non-decreasing in each argument, it follows that $\phi(\mathbf{1}^A, \boldsymbol{x}^{C \setminus A})$ is non-decreasing in $x_i$ for all $i \in C \setminus A$.

Hence, $(C \setminus A, \phi')$ is indeed a binary monotone system.

# Exercise 2.6 (cont.)

We then let $i \in (C \setminus A)$. By assumption $i$ is relevant in $(C, \phi)$. That is, there exists a $(\cdot_i, \boldsymbol{x})$ such that:

$$\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x}) = 1.$$

This equation can also be written as:

$$\phi(1_i, \boldsymbol{x}^A, \boldsymbol{x}^{C \setminus A}) - \phi(0_i, \boldsymbol{x}^A, \boldsymbol{x}^{C \setminus A}) = 1.$$

Since all the components in $A$ are irrelevant in the original system, we may replace $\boldsymbol{x}^A$ by $\mathbf{1}^A$ without changing the value of $\phi$:

$$\phi(1_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A}) - \phi(0_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A}) = 1.$$

Hence, we have shown that there exists a vector $(\cdot_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A})$ such that:

$$\phi(1_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A}) - \phi(0_i, \mathbf{1}^A, \boldsymbol{x}^{C \setminus A}) = 1.$$

Thus, we conclude that $i$ is relevant in $(C \setminus A, \phi')$, and since this holds for all $i \in C \setminus A$, we conclude that $(C \setminus A, \phi')$ is coherent.

# Exercise 2.7

Let $(C, \phi)$ be a non-trivial binary monotone system of order $n$. Then for all $\boldsymbol{x} \in \{0, 1\}^n$ we have:

$$\prod_{i=1}^{n} x_i \leq \phi(\boldsymbol{x}) \leq \coprod_{i=1}^{n} x_i.$$

Prove the right-hand inequality.

SOLUTION: If $\coprod_{i=1}^{n} x_i = 1$, this inequality is trivial since $\phi(\boldsymbol{x}) \in \{0, 1\}$ for all $\boldsymbol{x} \in \{0, 1\}^n$.

If on the other hand $\coprod_{i=1}^{n} x_i = 0$, we must have $\boldsymbol{x} = \mathbf{0}$.

Since $(C, \phi)$ is assumed to be non-trivial, it follows that $\phi(\mathbf{0}) = 0$. Thus, the inequality is valid in this case as well.

This completes the proof of the right-hand inequality.

# Exercise 2.8

Let $(C, \phi)$ be a binary monotone system of order $n$. Show that for all $\boldsymbol{x}, \boldsymbol{y} \in \{0, 1\}^n$ we have:

$$\phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y})$$

Moreover, assume that $(C, \phi)$ is coherent. Prove that equality holds for all $\boldsymbol{x}, \boldsymbol{y} \in \{0, 1\}^n$ if and only if $(C, \phi)$ is a series system.

SOLUTION: Since $\boldsymbol{x} \cdot \boldsymbol{y} \leq \boldsymbol{x}$ and $\boldsymbol{x} \cdot \boldsymbol{y} \leq \boldsymbol{y}$ and $\phi$ is non-decreasing in each argument, we have:

$$\phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \phi(\boldsymbol{x}) \text{ and } \phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \phi(\boldsymbol{y}).$$

Hence, we have:

$$\phi(\boldsymbol{x} \cdot \boldsymbol{y}) \leq \min\{\phi(\boldsymbol{x}), \phi(\boldsymbol{y})\} = \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y}).$$

# Exercise 2.8 (cont.)

It remains to prove that if $(C, \phi)$ is coherent, then $\phi(\boldsymbol{x} \cdot \boldsymbol{y}) = \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y})$ if and only if $(C, \phi)$ is a series system.

SOLUTION: Assume first that $(C, \phi)$ is a series system. Then:

$$\phi(\boldsymbol{x} \cdot \boldsymbol{y}) = \prod_{i=1}^{n}(x_i \cdot y_i) = [\prod_{i=1}^{n} x_i] \cdot [\prod_{i=1}^{n} y_i] = \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y}).$$

Assume conversely that $\phi(\boldsymbol{x} \cdot \boldsymbol{y}) = \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{y})$, and choose $i \in C$ arbitrarily.

Since $(C, \phi)$ is coherent, there exists a vector $(\cdot_i, \boldsymbol{x})$ such that:

$$\phi(1_i, \boldsymbol{x}) = 1 \text{ and } \phi(0_i, \boldsymbol{x}) = 0.$$

# Exercise 2.8 (cont.)

For this particular $(\cdot_i, \boldsymbol{x})$ we have by the assumption that:

$$\begin{aligned}0 = \phi(0_i, \boldsymbol{x}) &= \phi((0_i, \mathbf{1}) \cdot (1_i, \boldsymbol{x})) \\ &= \phi(0_i, \mathbf{1}) \cdot \phi(1_i, \boldsymbol{x}) = \phi(0_i, \mathbf{1}) \cdot 1\end{aligned}$$

Hence, $\phi(0_i, \mathbf{1}) = 0$, and since obviously $\phi(1_i, \mathbf{1}) = 1$, we conclude that:

$$\phi(x_i, \mathbf{1}) = x_i, \text{ for } x_i = 0, 1.$$

Since $i \in C$ was chosen arbitrarily, we must have:

$$\phi(x_i, \mathbf{1}) = x_i, \text{ for } x_i = 0, 1, \text{ for all } i \in C.$$

# Exercise 2.8 (cont.)

By repeated use of the assumption, we get:

$$\begin{aligned}
\phi(\boldsymbol{x}) &= \phi((x_1, \mathbf{1}) \cdot (x_2, \mathbf{1}) \cdots (x_n, \mathbf{1})) \\
&= \phi(x_1, \mathbf{1}) \cdot \phi(x_2, \mathbf{1}) \cdots \phi(x_n, \mathbf{1}) \\
&= x_1 \cdot x_2 \cdots x_n = \prod_{i=1}^{n} x_i
\end{aligned}$$

Thus, we conclude that $(C, \phi)$ is a series system.

# Exercise 2.9

Prove that the dual system of a $k$-out-of-$n$ system is an $(n - k + 1)$-out-of-$n$ system.

SOLUTION: Assume that $(C, \phi)$ is a $k$-out-of-$n$-system. The structure function, $\phi$, can then be written as:

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{otherwise.} \end{cases}$$

More compactly we may write this as:

$$\phi(\boldsymbol{x}) = I(\sum_{i=1}^{n} x_i \geq k).$$

# Exercise 2.9 (cont.)

We then have:

$$\begin{aligned}
\phi^D(\boldsymbol{y}) &= 1 - \phi(\boldsymbol{1} - \boldsymbol{y}) = 1 - I(\sum_{i=1}^{n}(1 - y_i) \geq k) \\
&= 1 - I(n - \sum_{i=1}^{n} y_i \geq k) = 1 - I(\sum_{i=1}^{n} y_i \leq n - k) \\
&= I(\sum_{i=1}^{n} y_i > n - k) = I(\sum_{i=1}^{n} y_i \geq n - k + 1)
\end{aligned}$$

Hence, $(C^D, \phi^D)$ is an $(n - k + 1)$-out-of-$n$ system.

# Exercise 2.10

Let $S$ be a stochastic variable with values in $\{0, 1, \ldots, n\}$. We then define the *generating function* of $S$ as:

$$G_S(y) = E[y^S] = \sum_{s=0}^{n} y^s P(S = s).$$

a) Explain why $G_S(y)$ is a polynomial, and give an interpretation of the coefficients of this polynomial.

SOLUTION: $G_S(y)$ is a polynomial because all the terms in the sum are of the form $a_s y^s$, $s = 0, 1, \ldots, n$.

The coefficient $a_s$ is equal to $P(S = s)$.

# Exercise 2.10 (cont.)

NOTE: A polynomial $g(y) = \sum_{s=0}^{n} a_s y^n$ is a generating function for a random variable $S$ with values in $\{0, 1, \dots n\}$ if and only if:

$$a_s \geq 0, \quad s = 0, 1, \dots, n$$
$$\sum_{s=0}^{n} a_s = 1.$$

# Exercise 2.10 (cont.)

b) Let $T$ be another non-negative integer valued stochastic variable with values in $\{0, 1, \dots, m\}$ which is independent of $S$. Show that:

$$G_{S+T}(y) = G_S(y) \cdot G_T(y).$$

SOLUTION: By the definition of a generating function and the independence of $S$ and $T$ we have:

$$\begin{aligned} G_{S+T}(y) &= E[y^{S+T}] = E[y^S \cdot y^T] \\ &= E[y^S] \cdot E[y^T] \text{ (using that } S \text{ and } T \text{ are independent)} \\ &= G_S(y) \cdot G_T(y) \end{aligned}$$

# Exercise 2.10 (cont.)

c) Let $X_1, \ldots, X_n$ be independent binary variables with $P(X_i = 1) = p_i$ and $P(X_i = 0) = 1 - p_i = q_i$, $i = 1, \ldots, n$. Show that:

$$G_{X_i}(y) = q_i + p_i y, \quad i = 1, \ldots, n.$$

SOLUTION: By the definition of a generating function we get:

$$\begin{aligned} G_{X_i}(y) &= E[y^{X_i}] \\ &= y^0 \cdot P(X_i = 0) + y^1 P(X_i = 1) \\ &= q_i + p_i y, \qquad i = 1, \ldots, n. \end{aligned}$$

## Exercise 2.10 (cont.)

d) Introduce:

$$S_j = \sum_{i=1}^{j} X_i, \quad j = 1, 2, \ldots, n,$$

and assume that we have computed $G_{S_j}(y)$. Thus, all the coefficients of $G_{S_j}(y)$ are known at this stage. We then compute:

$$G_{S_{j+1}}(y) = G_{S_j}(y) \cdot G_{X_{j+1}}(y).$$

How many algebraic operations (addition and multiplication) will be needed to complete this task?

# Exercise 2.10 (cont.)

SOLUTION: Assume that we have computed $G_{S_j}(y)$, and that:

$$G_{S_j}(y) = a_{j0} + a_{j1}y + a_{j2}y^2 + \cdots + a_{jj}y^j.$$

Then:

$$\begin{aligned} G_{S_{j+1}}(y) &= G_{S_j}(y) \cdot G_{X_{j+1}}(y) \\ &= (a_{j0} + a_{j1}y + a_{j2}y^2 + \cdots + a_{jj}y^j) \cdot (q_{i+1} + p_{i+1}y) \end{aligned}$$

In order to compute $G_{S_{j+1}}(y)$ we need to do $2(j+1)$ multiplications and $j$ additions.

# Exercise 2.10 (cont.)

e) Explain how generating functions can be used in order to calculate the reliability of a $k$-out-of-$n$ system. What can you say about the order of this algorithm.

SOLUTION: In order to compute $G_S(y) = G_{S_n}(y)$, $2 \cdot (2 + 3 + \cdots + n)$ multiplications and $(1 + 2 + \cdots (n-1))$ additions are needed. Thus, the number of operations grows roughly proportionally to $n^2$ operations.

Having calculated $G_S(y)$, a polynomial of degree $n$, the distribution of $S$ is given by the coefficients of this polynomial.

If $(C, \phi)$ is a $k$-out-of-$n$-system with component state variables $X_1, \ldots, X_n$, then the reliability of this system is given by:

$$P(\phi(\boldsymbol{X}) = 1) = P(S \geq k) = \sum_{s=k}^{n} P(S = s)$$

Thus, the reliability of $(C, \phi)$ can be calculated in $O(n^2)$-time.