STK3405 – Exercises Chapter 3

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Exercise 3.1

Prove equation (3.2) in another way than what is done in the proof of Theorem 3.1.1.

We need to show that the reliability function of a monotone system where the component state variables are independent, satisfies the following:

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p}), \quad i = 1, 2, \ldots, n.$$

PROOF: By conditioning on the state of component i we have:

$$\begin{aligned} h(\boldsymbol{p}) &= E[\phi(\boldsymbol{X})] \\ &= E[\phi(\boldsymbol{X}) | X_i = 1] P(X_i = 1) + E[\phi(\boldsymbol{X}) | X_i = 0] P(X_i = 0) \\ &= p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p}). \end{aligned}$$

Exercise 3.2

Prove Theorem 3.2.5:

Theorem

Let (C, ϕ) be a binary monotone system, and let (C^D, ϕ^D) be its dual. Then the following statements hold:

- $\boldsymbol{x}$ *is a path vector (alternatively, cut vector) for* (C, ϕ) *if and only if* $\boldsymbol{x}^D$ *is a cut vector (path vector) for* (C^D, ϕ^D).
- *A minimal path set (alternatively, cut set) for* (C, ϕ) *is a minimal cut set (path set) for* (C^D, ϕ^D).

Exercise 3.2 (cont.)

PROOF: Assume that $\boldsymbol{x}$ is a *path vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 1$ and we get:

$$\phi^D(\boldsymbol{x}^D) = 1 - \phi(\boldsymbol{1} - \boldsymbol{x}^D) = 1 - \phi(\boldsymbol{x}) = 1 - 1 = 0.$$

Hence, $\boldsymbol{x}^D$ is a *cut vector* for (C^D, ϕ^D).

Similarly, assume that $\boldsymbol{x}$ is a *cut vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 0$ and we get:

$$\phi^D(\boldsymbol{x}^D) = 1 - \phi(\boldsymbol{1} - \boldsymbol{x}^D) = 1 - \phi(\boldsymbol{x}) = 1 - 0 = 1.$$

Hence, $\boldsymbol{x}^D$ is a *path vector* for (C^D, ϕ^D).

Exercise 3.2 (cont.)

Assume that $\boldsymbol{x}$ is a *minimal path vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 1$ and $\phi(\boldsymbol{y}) = 0$ for all $\boldsymbol{y} < \boldsymbol{x}$. We have already shown that $\phi^D(\boldsymbol{x}^D) = 0$.

We then choose $\boldsymbol{y}^D > \boldsymbol{x}^D$, and note that this implies that $\boldsymbol{y} < \boldsymbol{x}$. Thus, we get: we have:

$$\phi^D(\boldsymbol{y}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{y}^D) = 1 - \phi(\boldsymbol{y}) = 1 - 0 = 1.$$

Hence, $\boldsymbol{x}^D$ is a *minimal cut vector* for (C^D, ϕ^D).

Similarly, assume that $\boldsymbol{x}$ is a *minimal cut vector* for (C, ϕ). Then by definition $\phi(\boldsymbol{x}) = 0$ and $\phi(\boldsymbol{y}) = 1$ for all $\boldsymbol{y} > \boldsymbol{x}$. We have already shown that $\phi^D(\boldsymbol{x}^D) = 1$.

We then choose $\boldsymbol{y}^D < \boldsymbol{x}^D$, and note that this implies that $\boldsymbol{y} > \boldsymbol{x}$. Thus, we get:

$$\phi^D(\boldsymbol{y}^D) = 1 - \phi(\mathbf{1} - \boldsymbol{y}^D) = 1 - \phi(\boldsymbol{y}) = 1 - 1 = 0.$$

Hence, $\boldsymbol{x}^D$ is a *minimal path vector* for (C^D, ϕ^D).

Exercise 3.2 (cont.)

Assume that $P \subseteq C$ is a *minimal path set* for (C, ϕ), and let $\boldsymbol{x}$ be the corresponding *minimal path vector*.

We have already shown that this implies that $\boldsymbol{x}^D = \mathbf{1} - \boldsymbol{x}$ is a minimal cut vector for (C^D, ϕ^D). Hence, it follows that:

$$K^D = \{i : x_i^D = 0\} = \{i : x_i = 1\} = P$$

is a *minimal cut set* for (C^D, ϕ^D).

Assume that $K \subseteq C$ is a *minimal cut set* for (C, ϕ), and let $\boldsymbol{x}$ be the corresponding *minimal cut vector*.

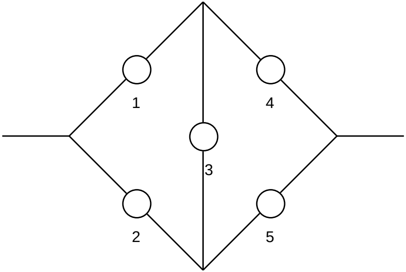
We have already shown that this implies that $\boldsymbol{x}^D = \mathbf{1} - \boldsymbol{x}$ is a minimal path vector for (C^D, ϕ^D). Hence, it follows that:

$$P^D = \{i : x_i^D = 1\} = \{i : x_i = 0\} = K$$

is a *minimal path set* for (C^D, ϕ^D).

Exercise 3.3

Find the minimal path and minimal cut sets of the brigde structure:



Minimal path sets:

$P_1 = \{1, 4\}$, $P_2 = \{1, 3, 5\}$, $P_3 = \{2, 3, 4\}$, $P_4 = \{2, 5\}$.

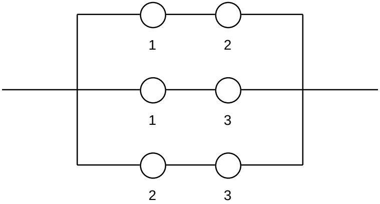
Minimal cut sets:

$K_1 = \{1, 2\}$, $K_2 = \{1, 3, 5\}$, $K_3 = \{2, 3, 4\}$, $K_4 = \{4, 5\}$.

Exercise 3.4

Find the representations via the minimal path sets and the minimal cut sets:

(i) **2-out-3 system:**



Minimal path sets: $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$, $P_3 = \{2, 3\}$.

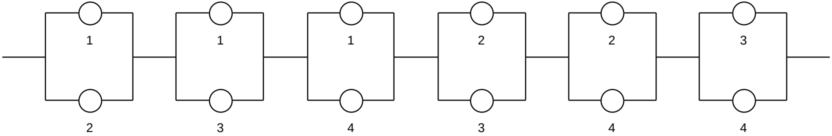
$$\phi(\boldsymbol{X}) = (X_1 \cdot X_2) \amalg (X_1 \cdot X_3) \amalg (X_2 \cdot X_3)$$

Minimal cut sets: $K_1 = \{1, 2\}$, $K_2 = \{1, 3\}$, $K_3 = \{2, 3\}$.

$$\phi(\boldsymbol{X}) = (X_1 \amalg X_2) \cdot (X_1 \amalg X_3) \cdot (X_2 \amalg X_3)$$

Exercise 3.4 (cont.)

(ii) **3-out-4 system:**



Minimal path sets:
$P_1 = \{1, 2, 3\}$, $P_2 = \{1, 2, 4\}$, $P_3 = \{1, 3, 4\}$, $P_4 = \{2, 3, 4\}$.

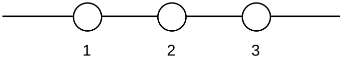
$$\phi(\boldsymbol{X}) = (X_1 \cdot X_2 \cdot X_3) \amalg (X_1 \cdot X_2 \cdot X_4) \amalg (X_1 \cdot X_3 \cdot X_4) \amalg (X_2 \cdot X_3 \cdot X_4)$$

Minimal cut sets:
$K_1 = \{1, 2\}$, $K_2 = \{1, 3\}$, $K_3 = \{1, 4\}$, $K_4 = \{2, 3\}$, $K_5 = \{2, 4\}$, $K_6 = \{3, 4\}$.

$$\phi(\boldsymbol{X}) = (X_1 \amalg X_2) \cdot (X_1 \amalg X_3) \cdot (X_1 \amalg X_4) \cdot (X_2 \amalg X_3) \cdot (X_2 \amalg X_4) \cdot (X_3 \amalg X_4)$$

Exercise 3.4 (cont.)

(iii) **Series system of 3 components:**



Minimal path set: $P = \{1, 2, 3\}$.

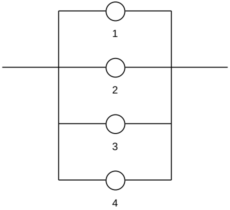
$$\phi(\boldsymbol{X}) = X_1 \cdot X_2 \cdot X_3$$

Minimal cut sets: $K_1 = \{1\}$, $K_2 = \{2\}$, $K_3 = \{3\}$.

$$\phi(\boldsymbol{X}) = X_1 \cdot X_2 \cdot X_3$$

Exercise 3.4 (cont.)

(iv) **Parallel system of 4 components:**



Minimal path set: $P_1 = \{1\}$, $P_2 = \{2\}$, $P_3 = \{3\}$, $P_4 = \{4\}$.

$$\phi(\boldsymbol{X}) = X_1 \amalg X_2 \amalg X_3 \amalg X_4$$

Minimal cut sets: $K = \{1, 2, 3, 4\}$.

$$\phi(\boldsymbol{X}) = X_1 \amalg X_2 \amalg X_3 \amalg X_4$$

Exercise 3.5

Consider a coherent system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$. Prove that

$$\bigcup_{j=1}^{p} P_j = \bigcup_{j=1}^{k} K_j = C.$$

NOTE: Since all the minimal path and cut sets are subsets of the component set C, we obviously have that:

$$\bigcup_{j=1}^{p} P_j \subseteq C \text{ and } \bigcup_{j=1}^{k} K_j \subseteq C$$

In order to show that for coherent systems the three sets are equal we show a slightly more general result.

Exercise 3.5 (cont.)

Theorem

Let (C, ϕ) be a binary monotone system. Then the following three statements are equivalent:

- *$i \in C$ is relevant*
- *$i \in P$ for at least one minimal path set P*
- *$i \in K$ for at least one minimal cut set K*

PROOF: Assume that $i \in C$ is relevant. Then there exists a vector $(\cdot_i, \boldsymbol{x})$ such that:

$$\phi(1_i, \boldsymbol{x}) = 1 \quad \text{and} \quad \phi(0_i, \boldsymbol{x}) = 0 \qquad (*)$$

NOTE: We can always choose $(\cdot_i, \boldsymbol{x})$ such that it is a *minimal* vector with the property $(*)$ in the sense that if $(\cdot_i, \boldsymbol{y}) < (\cdot_i, \boldsymbol{x})$, then $\phi(1_i, \boldsymbol{y}) = \phi(0_i, \boldsymbol{y}) = 0$.

Exercise 3.5 (cont.)

It then follows that $(1_i, \boldsymbol{x})$ is a minimal path vector since $\boldsymbol{y} < (1_i, \boldsymbol{x})$ implies that $\phi(\boldsymbol{y}) = 0$.

We then let $P = \{j \in C : x_j = 1\}$, which by definition this is a minimal path set. Moreover, $i \in P$.

Thus, we have shown that if $i \in C$ is relevant, then $i \in P$ for at least one minimal path set P.

We now show the converse implication, i.e., that if $i \in P$ for at least one minimal path set P, then $i \in C$ is relevant.

Exercise 3.5 (cont.)

Assume that there exists a minimal path set P such that $i \in P$, and let $\boldsymbol{x}$ be the corresponding minimal path vector. Then by definition:

$$x_i = 1 \quad \text{and} \quad \phi(\boldsymbol{x}) = \phi(1_i, \boldsymbol{x}) = 1.$$

Moreover, if $\boldsymbol{y} < \boldsymbol{x}$, then $\phi(\boldsymbol{y}) = 0$. Since in particular $(0_i, \boldsymbol{x}) < \boldsymbol{x}$, it follows that $\phi(0_i, \boldsymbol{x}) = 0$, i.e., i is relevant.

Thus, we have shown that if $i \in P$ for at least one minimal path set P, then $i \in C$ is relevant.

The proof that a component $i \in C$ is relevant if and only if $i \in K$ for at least one minimal cut set K is proved in a similar way.

Exercise 3.5 (cont.)

As direct consequence of the theorem we just proved we get:

Corollary

Consider a coherent system (C, ϕ) with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$. We then have:

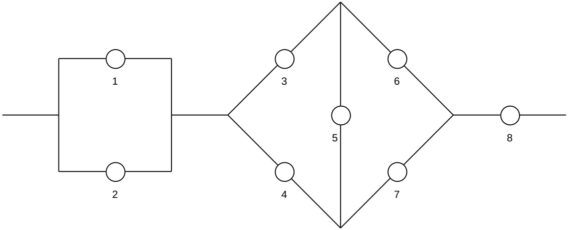
$$\bigcup_{j=1}^{p} P_j = \bigcup_{j=1}^{k} K_j = C.$$

PROOF: If (C, ϕ) is coherent, then *all* components are relevant. Hence, every component is contained in at least one minimal path set and in at least one minimal cut set.

Hence, the three sets must be equal.

Exercise 3.6

Find the minimal path sets and the minimal cut sets of the system in the system below.



Minimal path sets:

$P_1 = \{1,3,6,8\}$, $P_2 = \{1,3,5,7,8\}$, $P_3 = \{1,4,5,6,8\}$, $P_4 = \{1,4,7,8\}$,
$P_5 = \{2,3,6,8\}$, $P_6 = \{2,3,5,7,8\}$, $P_7 = \{2,4,5,6,8\}$, $P_8 = \{2,4,7,8\}$.

Minimal cut sets:

$K_1 = \{1,2\}$, $K_2 = \{3,4\}$, $K_3 = \{3,5,7\}$, $K_4 = \{4,5,6\}$, $K_5 = \{6,7\}$, $K_6 = \{8\}$.

Exercise 3.6 (cont.)

Find two different expressions for the structure function of this system.

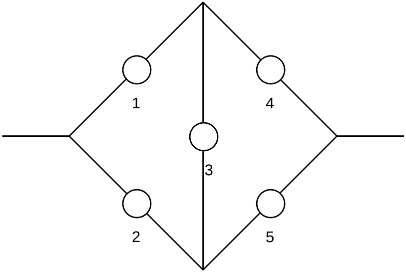
$$\phi(\boldsymbol{X}) = \coprod_{j=1}^{8} \prod_{i \in P_j} X_i$$

$$\phi(\boldsymbol{X}) = \prod_{j=1}^{6} \coprod_{i \in K_j} X_i$$

$$\phi(\boldsymbol{X}) = (X_1 \amalg X_2) \cdot [X_5 \cdot (X_3 \amalg X_4) \cdot (X_6 \amalg X_7) + (1 - X_5) \cdot ((X_3 X_6) \amalg (X_4 X_7))] \cdot X_8$$

Exercise 3.7

Show that if (C, ϕ) is a bridge structure, then (C^D, ϕ^D) is a bridge structure as well.



Minimal path sets:

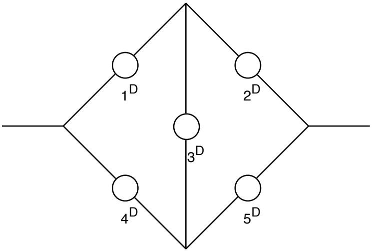
$P_1 = \{1, 4\}$, $P_2 = \{1, 3, 5\}$, $P_3 = \{2, 3, 4\}$, $P_4 = \{2, 5\}$

Minimal cut sets:

$K_1 = \{1, 2\}$, $K_2 = \{1, 3, 5\}$, $K_3 = \{2, 3, 4\}$, $K_4 = \{4, 5\}$.

Exercise 3.7 (cont.)

The dual system (C^D, ϕ^D):



Minimal path sets:

$P_1 = \{1^D, 2^D\}$, $P_2 = \{1^D, 3^D, 5^D\}$, $P_3 = \{2^D, 3^D, 4^D\}$, $P_4 = \{4^D, 5^D\}$

Minimal cut sets:

$K_1 = \{1^D, 4^D\}$, $K_2 = \{1^D, 3^D, 5^D\}$, $K_3 = \{2^D, 3^D, 4^D\}$, $K_4 = \{2^D, 5^D\}$.

Exercise 3.8

Let (A, χ) be a module of (C, ϕ). Assume that $\boldsymbol{x}_1$ and $\boldsymbol{x}_0$ are such that $\chi(\boldsymbol{x}_1^A) = 1$ and $\chi(\boldsymbol{x}_0^A) = 0$. Prove that for all $(\cdot^A, \boldsymbol{x}^{\bar{A}})$ we have:

$$\phi(\boldsymbol{x}_1^A, \boldsymbol{x}_1^{\bar{A}}) = \phi(\boldsymbol{1}^A, \boldsymbol{x}_1^{\bar{A}}) \quad \text{and} \quad \phi(\boldsymbol{x}_0^A, \boldsymbol{x}_0^{\bar{A}}) = \phi(\boldsymbol{0}^A, \boldsymbol{x}_0^{\bar{A}}).$$

PROOF: Let ψ be the organising structure function for ϕ and χ. That is, we have:

$$\phi(\boldsymbol{x}) = \psi(\chi(\boldsymbol{x}^A), \boldsymbol{x}^{\bar{A}}) \quad \text{for all } \boldsymbol{x}.$$

Hence, since χ is non-decreasing we have by the assumptions that:

$$\begin{aligned}\phi(\boldsymbol{x}_1) &= \psi(\chi(\boldsymbol{x}_1^A), \boldsymbol{x}_1^{\bar{A}}) = \psi(1, \boldsymbol{x}_1^{\bar{A}}) \\ &= \psi(\chi(\boldsymbol{1}^A), \boldsymbol{x}_1^{\bar{A}}) = \phi(\boldsymbol{1}^A, \boldsymbol{x}_1^{\bar{A}}).\end{aligned}$$

Similarly:

$$\begin{aligned}\phi(\boldsymbol{x}_0) &= \psi(\chi(\boldsymbol{x}_0^A), \boldsymbol{x}_0^{\bar{A}}) = \psi(0, \boldsymbol{x}_0^{\bar{A}}) \\ &= \psi(\chi(\boldsymbol{0}^A), \boldsymbol{x}_0^{\bar{A}}) = \phi(\boldsymbol{0}^A, \boldsymbol{x}_0^{\bar{A}}).\end{aligned}$$

Exercise 3.9

Find all the modules of the following structure function:

$$\phi(\boldsymbol{x}) = (x_1 \cdot (x_2 \amalg x_3)) \amalg (x_4 \amalg x_5).$$

SOLUTION: (Only non-trivial modules are included here)

- $A_1 = \{2, 3\}, \quad \chi_1(\boldsymbol{x}^{A_1}) = x_2 \amalg x_3, \quad \psi_1(\chi_1, \boldsymbol{x}^{\bar{A}_1}) = (x_1 \cdot \chi_1) \amalg (x_4 \amalg x_5)$
- $A_2 = \{1, 2, 3\}, \quad \chi_2(\boldsymbol{x}^{A_2}) = x_1(x_2 \amalg x_3), \quad \psi_2(\chi_2, \boldsymbol{x}^{\bar{A}_2}) = \chi_2 \amalg (x_4 \amalg x_5)$
- $A_3 = \{4, 5\}, \quad \chi_3(\boldsymbol{x}^{A_3}) = x_4 \amalg x_5, \quad \psi_3(\chi_3, \boldsymbol{x}^{\bar{A}_3}) = (x_1(x_2 \amalg x_3)) \amalg \chi_3$
- $A_4 = \{1, 2, 3, 4\}, \quad \chi_4(\boldsymbol{x}^{A_4}) = (x_1(x_2 \amalg x_3)) \amalg x_4, \quad \psi_4(\chi_4, \boldsymbol{x}^{\bar{A}_4}) = \chi_4 \amalg x_5$
- $A_4 = \{1, 2, 3, 5\}, \quad \chi_5(\boldsymbol{x}^{A_5}) = (x_1(x_2 \amalg x_3)) \amalg x_5, \quad \psi_5(\chi_5, \boldsymbol{x}^{\bar{A}_5}) = \chi_5 \amalg x_4$

Exercise 3.10

Let (C, ϕ) be a k-out-of-n system where $1 < k < n$, and assume that (A, χ) is a module of (C, ϕ) such that $1 < |A| < n$.

We can then find a minimal path set P (i.e., a set where $|P| = k$) such that:

$$(A \setminus P) \neq \emptyset \quad \text{and} \quad (A \cap P) \neq \emptyset \quad \text{and} \quad (P \setminus A) \neq \emptyset.$$

To see this, we note that since $1 < |A| < n$, there exists at least two components $i_1, i_2 \in A$, and a third component $i_3 \notin A$.

Since $1 < k < n$, the set $\{i_2, i_3\}$ can be extended to a minimal path set P, i.e., a set of size k, and such that $i_1 \notin P$. It then follows that:

$$\begin{aligned}
i_1 \in (A \setminus P) \quad &\Rightarrow \quad (A \setminus P) \neq \emptyset \\
i_2 \in (A \cap P) \quad &\Rightarrow \quad (A \cap P) \neq \emptyset \\
i_3 \in (P \setminus A) \quad &\Rightarrow \quad (P \setminus A) \neq \emptyset
\end{aligned}$$

Exercise 3.10

By using that:

$$A \setminus P \neq \emptyset \quad \text{and} \quad A \cap P \neq \emptyset \quad \text{and} \quad P \setminus A \neq \emptyset,$$

it follows that:

$$\phi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{P \setminus A}, \mathbf{0}) = 1,$$
$$\phi(\mathbf{0}^{A \setminus P}, \mathbf{0}^{A \cap P}, \mathbf{1}^{P \setminus A}, \mathbf{0}) = 0.$$

We now try to determine the value of $\chi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P})$, and consider two cases:

CASE 1. $\chi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}) = 0$. By Exercise 3.8 this implies that:

$$1 = \phi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{P \setminus A}, \mathbf{0}) = \phi(\mathbf{0}^{A \setminus P}, \mathbf{0}^{A \cap P}, \mathbf{1}^{P \setminus A}, \mathbf{0}) = 0.$$

That is, we have arrived at a contradiction.

Exercise 3.10 (cont.)

Since $i_3 \in (P \setminus A)$, it follows that:

$$|(A \cap P) \cup ((P \setminus A) \setminus i_3)| = |P \setminus i_3| = k - 1$$

Hence, we have:

$$\phi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{(P \setminus A) \setminus i_3}, \mathbf{0}) = 0.$$

Since $i_1 \in (A \setminus P)$, it follows that:

$$|(A \setminus P) \cup (A \cap P) \cup ((P \setminus A) \setminus i_3)| \geq |i_1 \cup (P \setminus i_3)| = k$$

Hence, we have:

$$\phi(\mathbf{1}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{(P \setminus A) \setminus i_3}, \mathbf{0}) = 1.$$

CASE 2. $\chi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}) = 1$. By Exercise 3.8 this implies that:

$$0 = \phi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{(P \setminus A) \setminus i_3}, \mathbf{0}) = \phi(\mathbf{1}^{A \setminus P}, \mathbf{1}^{A \cap P}, \mathbf{1}^{(P \setminus A) \setminus i_3}, \mathbf{0}) = 1.$$

That is, we have arrived at a contradiction.

Exercise 3.10 (cont.)

Since both the cases, $\chi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}) = 0$ and $\chi(\mathbf{0}^{A \setminus P}, \mathbf{1}^{A \cap P}) = 1$, lead to contradictions, we conclude that it is not possible to find any binary function $\chi(\boldsymbol{x}^A)$ such that (A, χ) is a module of (C, ϕ).

Hence, A cannot be a modular set of (C, ϕ).

Since this is true for all sets $A \subseteq C$ such that $1 < |A| < n$, we conclude that a k-out-of-n system (C, ϕ) where $1 < k < n$ has *no* non-trivial modules.