

STK3405 - Exercise 3.11-3.15

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Exercise 3.11

Consider the system shown in the figure below.

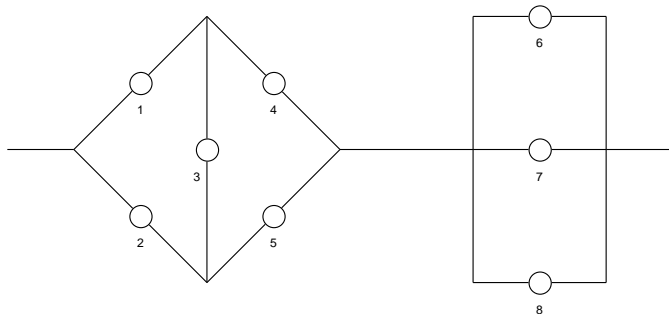


Figure: A system suited for modular decomposition.



Exercise 3.11 (cont.)

a) Find the structure function of the system by dividing the system into modules.

SOLUTION: We introduce the modules (A_1, χ_1) and (A_2, χ_2) where:

$$A_1 = \{1, 2, 3, 4, 5\}$$

$$\chi_1(\mathbf{x}^{A_1}) = x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))$$

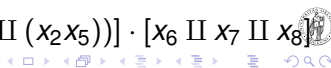
$$A_2 = \{6, 7, 8\}$$

$$\chi_2(\mathbf{x}^{A_2}) = x_6 \amalg x_7 \amalg x_8$$

Since the two modules are connected in series, we get that:

$$\phi(\mathbf{x}) = \chi_1(\mathbf{x}^{A_1}) \cdot \chi_2(\mathbf{x}^{A_2})$$

$$= [x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))] \cdot [x_6 \amalg x_7 \amalg x_8]$$



Exercise 3.11 (cont.)

b) Find the structure function of the system without dividing the system into modules.

SOLUTION: There are several ways of doing this. Here, we choose to pivot w.r.t. component 3 (i.e., the brigade):

$$\phi(\mathbf{x}) = x_3\phi(1_3, \mathbf{x}) + (1 - x_3)\phi(0_3, \mathbf{x})$$

Case 1. Component 3 is *functioning*.

In this case the first system becomes a series of two parallel systems consisting of 1, 2 and 4, 5. This is again in series with the parallel connection of 6, 7, 8. Hence we get:

$$\phi(1_3, \mathbf{x}) = (x_1 \text{ II } x_2) \cdot (x_4 \text{ II } x_5) \cdot (x_6 \text{ II } x_7 \text{ II } x_8)$$



Exercise 3.11 (cont.)

Case 2. Component 3 is *not functioning*.

In this case the first system becomes a parallel connection of the series connection of 1, 4 and the series connection of 2, 5. This is again in series with the parallel connection of 6, 7, 8. Hence we get:

$$\phi(\mathbf{0}_3, \mathbf{x}) = ((x_1 x_4) \amalg (x_2 x_5)) \cdot (x_6 \amalg x_7 \amalg x_8)$$

Combining the two cases we get:

$$\begin{aligned} \phi(\mathbf{x}) &= x_3(x_1 \amalg x_2) \cdot (x_4 \amalg x_5) \cdot (x_6 \amalg x_7 \amalg x_8) \\ &\quad + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5)) \cdot (x_6 \amalg x_7 \amalg x_8). \end{aligned}$$



Exercise 3.11 (cont.)

c) Compare the computational efforts in a) and b).

SOLUTION: In a) $\phi(\mathbf{x})$ is expressed as:

$$\phi(\mathbf{x}) = [x_3(x_1 \text{ II } x_2)(x_4 \text{ II } x_5) + (1 - x_3)((x_1 x_4) \text{ II } (x_2 x_5))] \cdot [x_6 \text{ II } x_7 \text{ II } x_8]$$

In b) $\phi(\mathbf{x})$ is expressed as:

$$\begin{aligned}\phi(\mathbf{x}) &= x_3(x_1 \text{ II } x_2) \cdot (x_4 \text{ II } x_5) \cdot (x_6 \text{ II } x_7 \text{ II } x_8) \\ &\quad + (1 - x_3)((x_1 x_4) \text{ II } (x_2 x_5)) \cdot (x_6 \text{ II } x_7 \text{ II } x_8).\end{aligned}$$

The formula obtained via the modular decomposition is easier to compute than the formula obtained with a direct approach since with the latter approach we need to compute $(x_6 \text{ II } x_7 \text{ II } x_8)$ twice.



Exercise 3.12

Consider the system shown in the figure below.

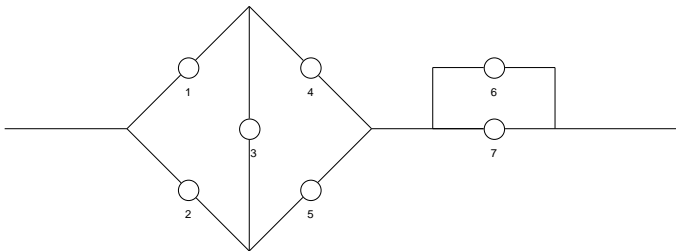


Figure: A mixed bridge and parallel system.

a) What is the structure function of this system?



Exercise 3.12 (cont.)

SOLUTION: We see that this is a bridge structure in series with a parallel structure.

Hence, by modular decomposition the structure function becomes:

$$\phi(\mathbf{x}) = \chi_1(\mathbf{x}^{A_1}) \cdot \chi_2(\mathbf{x}^{A_2})$$

where:

$$A_1 = \{1, 2, 3, 4, 5\}$$

$$\chi_1(\mathbf{x}^{A_1}) = x_3(x_1 \amalg x_2)(x_4 \amalg x_5) + (1 - x_3)((x_1 x_4) \amalg (x_2 x_5))$$

$$A_2 = \{6, 7\}$$

$$\chi_2(\mathbf{x}^{A_2}) = x_6 \amalg x_7$$



Exercise 3.12 (cont.)

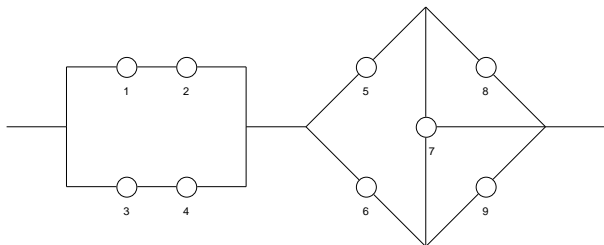
b) Assume that all of the component states are independent, and that $P(X_i = 1) = p_i$, $i = 1, \dots, 7$. What is the reliability function of the system?

SOLUTION: Using the independence we get:

$$\begin{aligned}h(\mathbf{p}) &= E[\phi(\mathbf{X})] \\&= E[[X_3(X_1 \text{ II } X_2)(X_4 \text{ II } X_5) + (1 - X_3)((X_1 X_4) \text{ II } (X_2 X_5))][X_6 \text{ II } X_7]] \\&= [E[X_3] \cdot E[X_1 \text{ II } X_2]E[X_4 \text{ II } X_5] + (1 - E[X_3]) \cdot E[(X_1 X_4) \text{ II } (X_2 X_5)]] \\&\quad \cdot E[X_6 \text{ II } X_7] \\&= [p_3 \cdot (p_1 \text{ II } p_2)(p_4 \text{ II } p_5) + (1 - p_3) \cdot ((p_1 p_4) \text{ II } (p_2 p_5))] \cdot (p_6 \text{ II } p_7)\end{aligned}$$



Exercise 3.13



NOTE: In this undirected network system component 7 is *not an edge*. Here component 7 should be interpreted as a *node*. This must be taken into account when we do a pivotal decomposition with respect to component 7.



Exercise 3.13 (cont.)

a) What is the structure function of this system?

SOLUTION: We start out by decomposing the system into two modules (A_1, χ_1) and (A_2, χ_2) where $A_1 = \{1, 2, 3, 4\}$ and $A_2 = \{5, 6, 7, 8, 9\}$.

Since the first module is a parallel connection of the series connection of 1, 2 and the series connection 3, 4, we get:

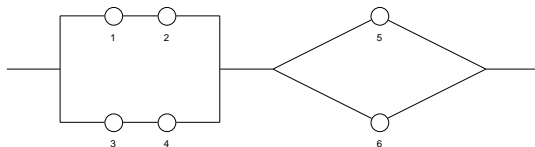
$$\chi_1(\mathbf{x}^{A_1}) = (x_1 x_2) \amalg (x_3 x_4)$$

To find $\chi_2(\mathbf{x}^{A_2})$, we do a pivotal decomposition w.r.t. component 7:

$$\chi_2(\mathbf{x}^{A_2}) = x_7 \chi_2(1_7, \mathbf{x}^{A_2}) + (1 - x_7) \chi_2(0_7, \mathbf{x}^{A_2})$$



Exercise 3.13 (cont.)



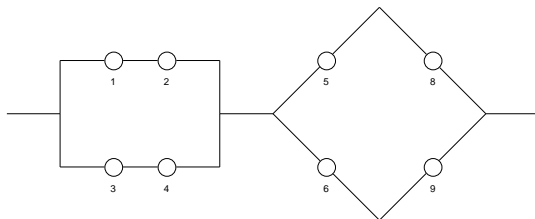
Case 1. Component 7 is *functioning*.

In this case components 8 and 9 become irrelevant and this module reduces to a parallel of components 5 and 6:

$$\chi_2(\mathbf{1}_7, \mathbf{x}^{A_2}) = x_5 \amalg x_6$$



Exercise 3.13 (cont.)



Case 2. Component 7 is *not functioning*.

In this case the module becomes a parallel connection of the series connection of 5 and 8 and the series connection of 6 and 9:

$$\chi_2(0_7, \mathbf{x}^{A_2}) = ((x_5 x_8) \amalg (x_6 x_9))$$



Exercise 3.13 (cont.)

Combining the two cases we get:

$$\begin{aligned}\chi_2(\mathbf{x}^{A_2}) &= x_7 \chi_2(\mathbf{1}_7, \mathbf{x}^{A_2}) + (1 - x_7) \chi_2(\mathbf{0}_7, \mathbf{x}^{A_2}) \\ &= x_7(x_5 \amalg x_6) + (1 - x_7)((x_5 x_8) \amalg (x_6 x_9))\end{aligned}$$



Exercise 3.13 (cont.)

Finally, since the two modules are in series we get:

$$\begin{aligned}\phi(\mathbf{x}) &= \chi_1(\mathbf{x}^{A_1}) \cdot \chi_2(\mathbf{x}^{A_2}) \\ &= [(x_1 x_2) \text{ II } (x_3 x_4)] \cdot [x_7(x_5 \text{ II } x_6) + (1 - x_7)((x_5 x_8) \text{ II } (x_6 x_9))]\end{aligned}$$

b) Assume that all of the component states are independent. What is the reliability function of the system?

SOLUTION: By a similar kind of calculation as in Exercise 3.12 b), using the independence assumption repeatedly, we find that:

$$\begin{aligned}h(\mathbf{p}) &= E[\chi_1(\mathbf{x}^{A_1})] \cdot E[\chi_2(\mathbf{x}^{A_2})] \\ &= E[(X_1 X_2) \text{ II } (X_3 X_4)] \cdot E[X_7(X_5 \text{ II } X_6) + (1 - X_7)((X_5 X_8) \text{ II } (X_6 X_9))] \\ &= [(p_1 p_2) \text{ II } (p_3 p_4)] \cdot [p_7(p_5 \text{ II } p_6) + (1 - p_7)((p_5 p_8) \text{ II } (p_6 p_9))]\end{aligned}$$



Exercise 3.14

Consider a binary monotone system (C, ϕ) with minimal cut sets K_1, \dots, K_k , where T_i denotes the lifetime of the i th component, $i = 1, \dots, n$, and where S denotes the lifetime of the system. Prove that:

$$S = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i$$

SOLUTION: Let U_j denote the lifetime of the j th minimal cut parallel structure. Then U_j equals the lifetime of the longest living component in K_j . That is, we have:

$$U_j = \max_{i \in K_j} T_i$$

Moreover, the lifetime of the system equals the lifetime of the shortest living minimal cut parallel structure. That is, we have:

$$S = \min_{1 \leq j \leq k} U_j = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i$$



Exercise 3.15

Consider a binary monotone system (C, ϕ) with minimal path sets P_1, \dots, P_p and minimal cut sets K_1, \dots, K_k and let f be a non-negative function defined over the positive integers $\{1, \dots, n\}$. Prove that

$$\min_{1 \leq j \leq k} \max_{i \in K_j} f(i) = \max_{1 \leq j \leq p} \min_{i \in P_j} f(i).$$

SOLUTION: Since f takes inputs in $C = \{1, \dots, n\}$ and is non-negative, $f(i)$ can be interpreted as the lifetime of the i th component, i.e., T_i . By Theorem 3.4.1 we know that:

$$\max_{1 \leq j \leq p} \min_{i \in P_j} T_i = \min_{1 \leq j \leq k} \max_{i \in K_j} T_i$$

Hence, by replacing T_i by $f(i)$ for all $i \in C$, we get:

$$\max_{1 \leq j \leq p} \min_{i \in P_j} f(i) = \min_{1 \leq j \leq k} \max_{i \in K_j} f(i)$$

which was what we wanted to prove.

