STK3405 - Exercise 4.1-4.8

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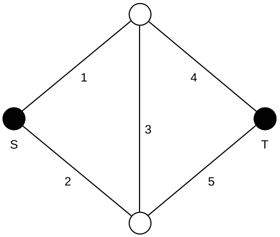
Exercise 4.1

Consider the bridge structure with independent component state variables and reliability function $h(\mathbf{p})$, and assume that $p_i = 0.9$, $i = 1, \ldots, 5$.



What do the bounds:

$$1 - \sum_{j=1}^{k} P(F_j) \leq h \leq \sum_{j=1}^{p} P(E_j)$$

reduce to in this case? Comment on the result.

Exercise 4.1 (cont.)

SOLUTION: The minimal path and minimal cut sets are:

$$P_1 = \{1,4\}, \quad P_2 = \{1,3,5\}, \quad P_3 = \{2,3,4\}, \quad P_4 = \{2,5\}$$

$$K_1 = \{1,2\}, \quad K_2 = \{1,3,5\}, \quad K_3 = \{2,3,4\}, \quad K_4 = \{4,5\}$$

Moreover, the reliability function is:

$$h(\mathbf{p}) = p_3(p_1 \amalg p_2)(p_4 \amalg p_5) + (1 - p_3)((p_1 p_4) \amalg (p_2 p_5)).$$

We then introduce:

$$E_j = \text{All the components in } P_j \text{ are functioning}, \quad j = 1,2,3,4$$
$$F_j = \text{All the components in } K_j \text{ are failed}, \quad j = 1,2,3,4$$

Exercise 4.1 (cont.)

Since $p_i = 0.9$, $i = 1, 2, 3, 4, 5$, we get:

$$P(E_1) = P(E_4) = 0.9^2, \quad P(E_2) = P(E_3) = 0.9^3$$

$$P(F_1) = P(F_4) = 0.1^2, \quad P(F_2) = P(F_3) = 0.1^3$$

Hence, the lower bound is given by:

$$1 - \sum_{j=1}^{4} P(F_j) = 1 - 2 \cdot 0.1^2 - 2 \cdot 0.1^3 = 0.978$$

and the upper bound is given by:

$$\sum_{j=1}^{4} P(E_j) = 2 \cdot 0.9^2 + 2 \cdot 0.9^3 = 3.078$$

Exercise 4.1 (cont.)

Finally, the exact reliability of the system is given by:

$$
\begin{aligned}
h(\mathbf{p}) &= p_3(p_1 \amalg p_2)(p_4 \amalg p_5) + (1 - p_3)((p_1 p_4) \amalg (p_2 p_5)) \\
&= 0.9 \cdot (0.9 \amalg 0.9)^2 + 0.1 \cdot (0.9^2 \amalg 0.9^2) \\
&= 0.9 \cdot (1 - (1 - 0.9)^2)^2 + 0.1 \cdot (1 - (1 - 0.9^2)^2) \\
&= 0.9 \cdot 0.99^2 + 0.1 \cdot (1 - 0.19^2) \\
&= 0.9 \cdot 0.9801 + 0.1 \cdot 0.9639 = 0.88209 + 0.09639 = 0.97848
\end{aligned}
$$

Summarizing this we get the bounds:

$$0.978 \leq 0.97848 \leq 3.078$$

The lower bound is very good, while the upper bound is very bad (even greater than 1). According to the compendium this holds in general when the component reliabilities are close to 1.

Exercise 4.2

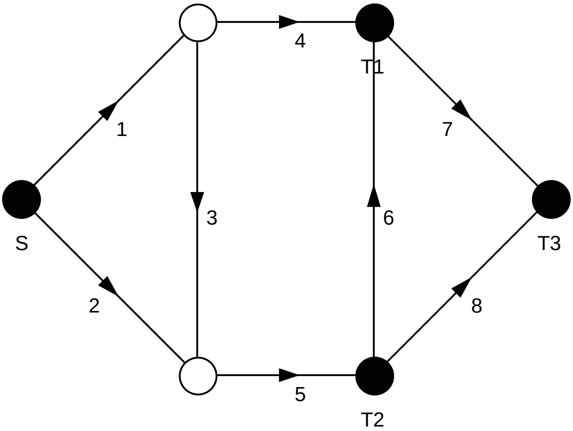
Draw an example of the following systems:

- An S3T system.
- An S5T system.

NOTE: This exercise can of course be solved in many different ways. The systems illustrated here are just some examples.

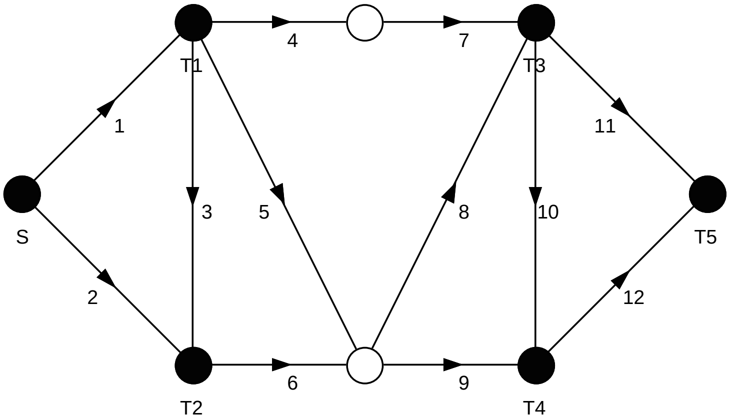
Exercise 4.2 (cont.)

An S3T system:



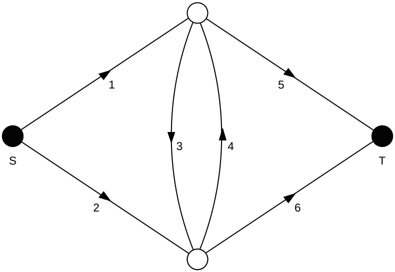
Exercise 4.2 (cont.)

An S5T system:



Exercise 4.3

Consider the S1T system below.



a) Find the minimal path sets of the system.

b) How many terms will there be in the inclusion-exclusion formula for the reliability of this system before simplification?

c) How many of these terms will vanish in the final simplified expression?

Exercise 4.3 (cont.)

SOLUTION:

a) The minimal path sets are:

$$P_1 = \{1, 5\}, P_2 = \{2, 6\}, P_3 = \{1, 3, 6\}, P_4 = \{2, 4, 5\}$$

b) The number of terms in the inclusion-exclusion formula based on the minimal path sets are (before cancelling terms):

$$2^p - 1 = 2^4 - 1 = 15.$$

Exercise 4.3 (cont.)

c) We introduce:

$$E_j = \text{All the components in } P_j \text{ are functioning}, \quad j = 1, 2, 3, 4.$$

From this it follows that:

$$P(E_{j_1} \cap \cdots \cap E_{j_r}) = P(\bigcap_{i \in P_{j_1} \cup \cdots \cup P_{j_r}} X_i = 1)$$

This implies that if:

$$P_{j_1} \cup \cdots \cup P_{j_r} = P_{k_1} \cup \cdots \cup P_{k_s},$$

then we have:

$$P(E_{j_1} \cap \cdots \cap E_{j_r}) = P(E_{k_1} \cap \cdots \cap E_{k_s})$$

Exercise 4.3 (cont.)

In this case we have that $C = \{1, 2, 3, 4, 5, 6\}$, and:

$$P_1 = \{1, 5\}, P_2 = \{2, 6\}, P_3 = \{1, 3, 6\}, P_4 = \{2, 4, 5\}$$

Hence, we get that:

$$P_3 \cup P_4 = P_1 \cup P_3 \cup P_4 = P_2 \cup P_3 \cup P_4 = P_1 \cup P_2 \cup P_3 \cup P_4 = C.$$

From this it follows that:

$$P(E_3 \cap E_4) = P(E_1 \cap E_3 \cap E_4) = P(E_2 \cap E_3 \cap E_4) = P(E_1 \cap E_2 \cap E_3 \cap E_4)$$

NOTE: C contains the *directed cycle* $\{3, 4\}$. Thus, according to the theory, the coefficient associated with this term is zero.

Exercise 4.3 (cont.)

$$
\begin{aligned}
h &= P(\bigcup_{j=1}^{4} E_j) \\
&= P(E_1) + P(E_2) + P(E_3) + P(E_4) \\
&- P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_1 \cap E_4) - P(E_2 \cap E_3) \\
&- P(E_2 \cap E_4) - P(E_3 \cap E_4) \\
&+ P(E_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_4) + P(E_1 \cap E_3 \cap E_4) \\
&+ P(E_2 \cap E_3 \cap E_4) \\
&- P(E_1 \cap E_2 \cap E_3 \cap E_4).
\end{aligned}
$$

Exercise 4.3 (cont.)

$$
\begin{aligned}
h = P(\bigcup_{j=1}^{4} E_j) \\
&= P(E_1) + P(E_2) + P(E_3) + P(E_4) \\
&- P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_1 \cap E_4) - P(E_2 \cap E_3) \\
&- P(E_2 \cap E_4) - P(E_3 \cap E_4) \\
&+ P(E_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_4) \\
&+ P(E_1 \cap E_3 \cap E_4) + P(E_2 \cap E_3 \cap E_4) \\
&- P(E_1 \cap E_2 \cap E_3 \cap E_4).
\end{aligned}
$$

Two terms, $P(E_1 \cap E_3 \cap E_4)$ and $P(E_2 \cap E_3 \cap E_4)$ correspond to odd formations of C, while $P(E_3 \cap E_4)$ and $P(E_1 \cap E_2 \cap E_3 \cap E_4)$ correspond to even formations of C. Since the number of odd formations is equal to the number of even formations, these terms are cancelled out.

Exercise 4.4

A linear consecutive k-out-of-n system is a binary monotone system (C, ϕ) with $P \subseteq C$ is a minimal path set if and only if P can be written as:

$$P = \{i, (i+1), \ldots, (i+k-1)\},$$

where $1 \leq k \leq n$, and

$$1 \leq i < i+k-1 \leq n.$$

Let (C, ϕ) be a linear consecutive 2-out-of-5 system. Show that (C, ϕ) cannot be an SKT-system.

[*Hint:* Let δ denote the signed domination function of the system. Compare $\delta(\{1,2,3,4\})$, $\delta(\{2,3,4,5\})$ and $\delta(\{1,2,3,4,5\})$ and interpret the result in the light of Theorem 4.4.3.]

Exercise 4.4 (cont.)

SOLUTION: By definition a linear consecutive k-out-of-n system it follows that the minimal path sets of (C, ϕ) are:

$$P_1 = \{1, 2\}, \quad P_2 = \{2, 3\}, \quad P_3 = \{3, 4\}, \quad P_4 = \{4, 5\}$$

From this we find the structure function:

$$\begin{aligned}
\phi(\mathbf{x}) &= 1 - (1 - x_1 x_2)(1 - x_2 x_3)(1 - x_3 x_4)(1 - x_4 x_5) \\
&= x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 \\
&\quad - x_1 x_2 x_3 - x_1 x_2 x_3 x_4 - x_1 x_2 x_4 x_5 - x_2 x_3 x_4 - x_2 x_3 x_4 x_5 - x_3 x_4 x_5 \\
&\quad + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_2 x_3 x_4 x_5 \\
&\quad - x_1 x_2 x_3 x_4 x_5
\end{aligned}$$

Exercise 4.4 (cont.)

SOLUTION: By definition a linear consecutive k-out-of-n system it follows that the minimal path sets of (C, ϕ) are:

$$P_1 = \{1, 2\}, \quad P_2 = \{2, 3\}, \quad P_3 = \{3, 4\}, \quad P_4 = \{4, 5\}$$

From this we find the structure function:

$$\begin{aligned}
\phi(\mathbf{x}) &= 1 - (1 - x_1 x_2)(1 - x_2 x_3)(1 - x_3 x_4)(1 - x_4 x_5) \\
&= x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 \\
&\quad - x_1 x_2 x_3 - x_1 x_2 x_3 x_4 - x_1 x_2 x_4 x_5 - x_2 x_3 x_4 - x_2 x_3 x_4 x_5 - x_3 x_4 x_5 \\
&\quad + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_4 x_5 + x_1 x_2 x_3 x_4 x_5 + x_2 x_3 x_4 x_5 \\
&\quad - x_1 x_2 x_3 x_4 x_5
\end{aligned}$$

Exercise 4.4 (cont.)

SOLUTION: By definition a linear consecutive k-out-of-n system it follows that the minimal path sets of (C, ϕ) are:

$$P_1 = \{1,2\}, \quad P_2 = \{2,3\}, \quad P_3 = \{3,4\}, \quad P_4 = \{4,5\}$$

From this we find the structure function:

$$\begin{aligned}
\phi(\mathbf{x}) &= 1 - (1 - x_1x_2)(1 - x_2x_3)(1 - x_3x_4)(1 - x_4x_5) \\
&= x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 \\
&\quad - x_1x_2x_3 - x_1x_2x_3x_4 - x_1x_2x_4x_5 - x_2x_3x_4 - x_2x_3x_4x_5 - x_3x_4x_5 \\
&\quad + x_1x_2x_3x_4 + x_1x_2x_3x_4x_5 + x_1x_2x_3x_4x_5 + x_2x_3x_4x_5 \\
&\quad - x_1x_2x_3x_4x_5 \\
&= x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 \\
&\quad - x_1x_2x_3 - x_1x_2x_4x_5 - x_2x_3x_4 - x_3x_4x_5 \\
&\quad + x_1x_2x_3x_4x_5
\end{aligned}$$

Exercise 4.4 (cont.)

We now consider the structure function:

$$\begin{aligned}\phi(\mathbf{x}) = {} & x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 \\ & - x_1x_2x_3 - x_1x_2x_4x_5 - x_2x_3x_4 - x_3x_4x_5 \\ & + x_1x_2x_3x_4x_5\end{aligned}$$

NOTE:

$$\delta(\{1,2,3,4\}) = 0 \quad \text{since } x_1x_2x_3x_4 \text{ does not occur in } \phi(\mathbf{x})$$
$$\delta(\{2,3,4,5\}) = 0 \quad \text{since } x_2x_3x_4x_5 \text{ does not occur in } \phi(\mathbf{x})$$

At the same time we see that $\delta(\{1,2,3,4,5\}) = 1$.

Exercise 4.4 (cont.)

If (C, ϕ) is an SKT-system, it follows from Theorem 4.4.3 that we must have:

$$\delta(A) = \begin{cases} (-1)^{|A|-v(A)+1} & \text{if } A \text{ is a union of minimal path sets,} \\ & \text{and } A \text{ does not contain a directed cycle} \\ 0 & \text{otherwise} \end{cases}$$

We then note that $\{1, 2, 3, 4\} = P_1 \cup P_3$, while $\delta(\{1, 2, 3, 4\}) = 0$. If (C, ϕ) is an SKT-system, this implies that set $\{1, 2, 3, 4\}$ must contain a directed cycle.

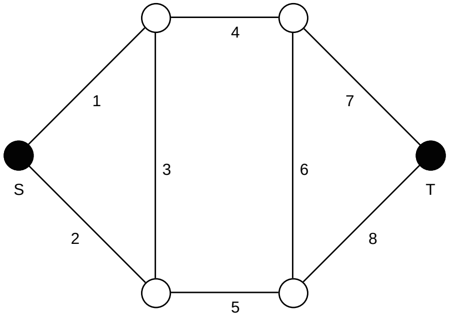
Similarly, $\{2, 3, 4, 5\} = P_2 \cup P_4$, while $\delta(\{2, 3, 4, 5\}) = 0$. If (C, ϕ) is an SKT-system, this implies that set $\{2, 3, 4, 5\}$ must contain a directed cycle.

However, still assuming that (C, ϕ) is an SKT-system, this implies that the larger set $\{1, 2, 3, 4, 5\}$ must contain a circuit as well, which contradicts that $\delta(\{1, 2, 3, 4, 5\}) = 1$.

Hence, we conclude that (C, ϕ) cannot be an SKT-system.

Exercise 4.5

Compute the reliability function of the undirected network system below which functions if and only if the nodes S and T can communicate through the network. Use the factoring algorithm.



Exercise 4.5 (cont.)

SOLUTION: We do pivotal decompositions w.r.t. the brigde components, 3 and 6:

$$
\begin{aligned}
h(\mathbf{p}) &= p_3 p_6 h_{+3+6}(\mathbf{p}) + (1-p_3) p_6 h_{-3+6}(\mathbf{p}) \\
&= + p_3 (1-p_6) h_{+3-6}(\mathbf{p}) + (1-p_3)(1-p_6) h_{-3-6}(\mathbf{p})
\end{aligned}
$$

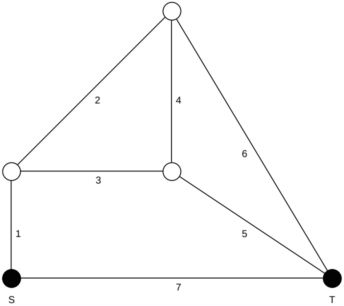
Using s-p-reductions on $h_{+3+6}(\mathbf{p})$, $h_{-3+6}(\mathbf{p})$, $h_{+3-6}(\mathbf{p})$ and $h_{-3-6}(\mathbf{p})$, we get:

$$
\begin{aligned}
h_{+3+6}(\mathbf{p}) &= (p_1 \amalg p_2) \cdot (p_4 \amalg p_5) \cdot (p_7 \amalg p_8) \\
h_{-3+6}(\mathbf{p}) &= [(p_1 p_4) \amalg (p_2 p_5)] \cdot (p_7 \amalg p_8) \\
h_{+3-6}(\mathbf{p}) &= (p_1 \amalg p_2) \cdot [(p_4 p_7) \amalg (p_5 p_8)] \\
h_{-3-6}(\mathbf{p}) &= [(p_1 p_4 p_7) \amalg (p_2 p_5 p_8)]
\end{aligned}
$$

The complete expression for the reliability function $h(\mathbf{p})$ is then obtained by inserting these function into the pivotal decomposition.

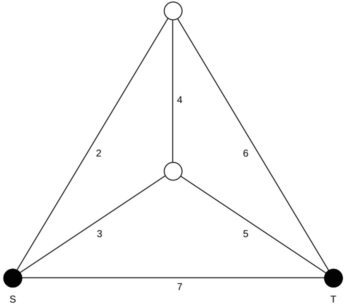
Exercise 4.6

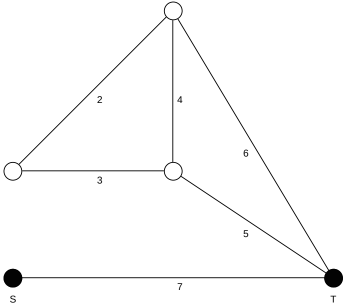
Compute the reliability function of the undirected network shown below by using the factoring algorithm (assuming independent component states).



Since the system is complex, we need to do a pivotal decomposition. Which component? Compare the computational work for different component choices.

Pivotal decomposition w.r.t. component 1:



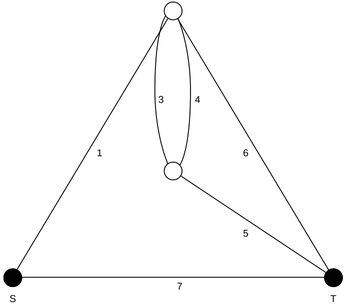


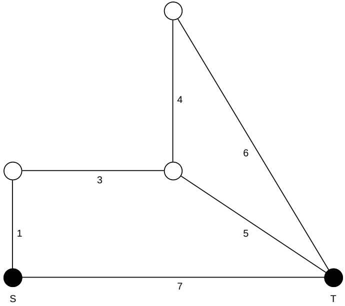
$$h_{+1}(\mathbf{p}) = [p_4 \cdot (p_2 \amalg p_3) \cdot (p_5 \amalg p_6) + (1 - p_4)((p_2 p_6) \amalg (p_3 p_5))] \amalg p_7$$

$$h_{-1}(\mathbf{p}) = p_7$$ incoherent system

Exercise 4.6 (cont.)

Pivotal decomposition w.r.t. component 2:



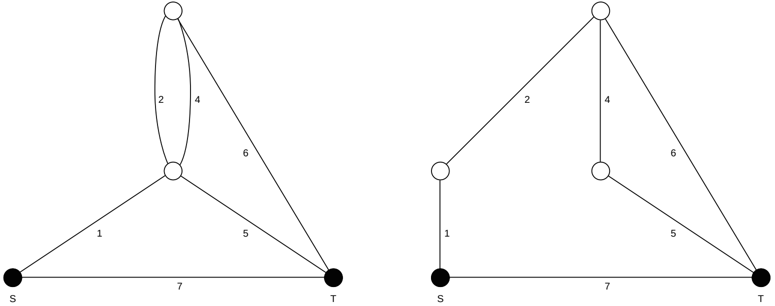


$$h_{+2}(\mathbf{p}) = [p_1 \cdot ([(p_3 \amalg p_4) \cdot p_5] \amalg p_6)] \amalg p_7$$

$$h_{-2}(\mathbf{p}) = [p_1 \cdot p_3 \cdot ((p_4 p_6) \amalg p_5)] \amalg p_7$$

Exercise 4.6 (cont.)

Pivotal decomposition w.r.t. component 3:

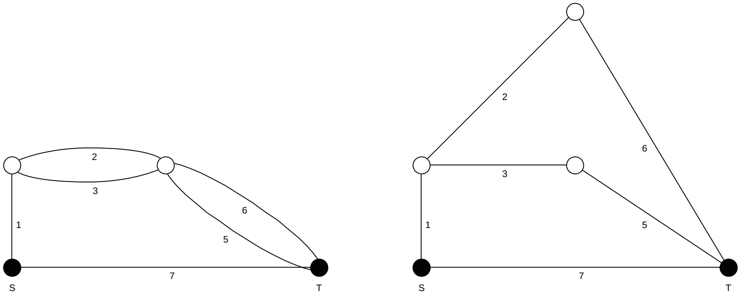


$$h_{+3}(\mathbf{p}) = [p_1 \cdot ([(p_2 \amalg p_4) \cdot p_6] \amalg p_5)] \amalg p_7$$

$$h_{-3}(\mathbf{p}) = [p_1 \cdot p_2 \cdot ((p_4 p_5) \amalg p_6)] \amalg p_7$$

Exercise 4.6 (cont.)

Pivotal decomposition w.r.t. component 4:

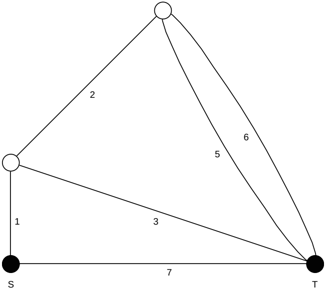


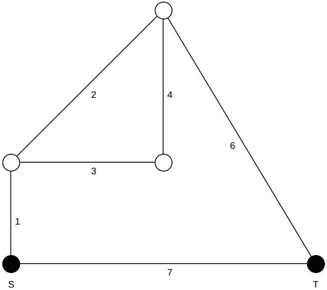
$$h_{+4}(\mathbf{p}) = [p_1 \cdot ((p_2 \amalg p_3) \cdot (p_5 \amalg p_6))] \amalg p_7$$

$$h_{-4}(\mathbf{p}) = [p_1 \cdot ((p_2 \cdot p_6) \amalg (p_3 \cdot p_5))] \amalg p_7$$

Exercise 4.6 (cont.)

Pivotal decomposition w.r.t. component 5:



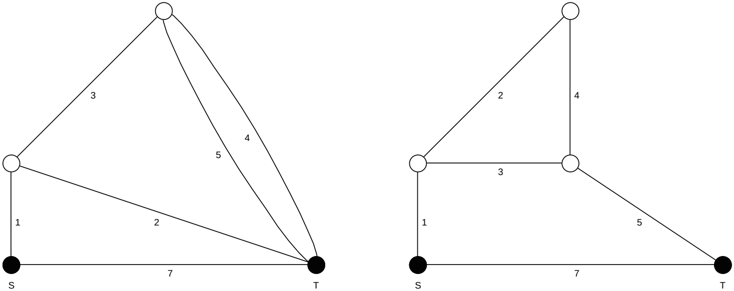


$$h_{+5}(\mathbf{p}) = [p_1 \cdot (p_3 \amalg [p_2 \cdot (p_5 \amalg p_6)])] \amalg p_7$$

$$h_{-5}(\mathbf{p}) = [p_1 \cdot (p_2 \amalg (p_3 p_4)) \cdot p_6] \amalg p_7$$

Exercise 4.6 (cont.)

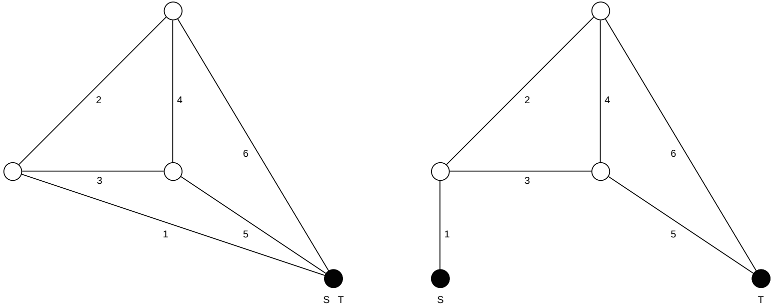
Pivotal decomposition w.r.t. component 6:



$$h_{+6}(\mathbf{p}) = [p_1 \cdot (p_2 \amalg [p_3 \cdot (p_4 \amalg p_5)])] \amalg p_7$$

$$h_{-6}(\mathbf{p}) = [p_1 \cdot (p_3 \amalg (p_2 p_4)) \cdot p_5] \amalg p_7$$

Pivotal decomposition w.r.t. component 7:



$h_{+7}(\mathbf{p}) = 1$ incoherent system

$h_{-7}(\mathbf{p}) = p_1 \cdot [p_4 \cdot (p_2 \amalg p_3) \cdot (p_5 \amalg p_6) + (1 - p_4)((p_2 p_6) \amalg (p_3 p_5))]$

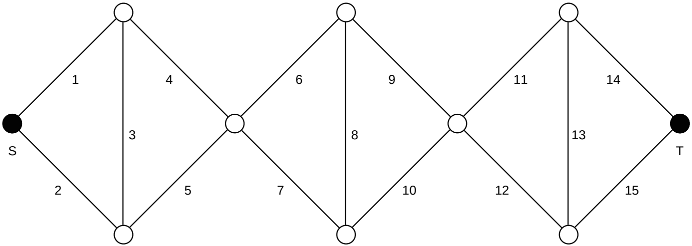
Exercise 4.6 (cont.)

If we do a pivotal decomposition w.r.t. components 1 or 7, one of the minors is incoherent, while the other is s-p-complex. Thus, another pivotal decomposition is needed.

If we do a pivotal decomposition w.r.t. any of the components 2, 3, 4, 5 or 6, both minors are coherent s-p-systems. Thus, no other pivotal decomposition is needed.

Exercise 4.7

A series connection of three bridge structures.



a) Compute the reliability function of the system by using the *factoring algorithm*.

Using the factoring algorithm we must do three pivotal decompositions, e.g., with respect to the bridge components 3, 8 and 13. Thus, we end up with an expression containing $2^3 = 8$ terms. We skip the details here.

Exercise 4.7 (cont.)

b) Use a modular decomposition $\{(A_1, \chi_1), (A_2, \chi_2), (A_3, \chi_3)\}$, where:

$$A_1 = \{1, 2, 3, 4, 5\}, \quad A_2 = \{6, 7, 8, 9, 10\}, \quad A_3 = \{11, 12, 13, 14, 15\},$$

$$\chi_1(\mathbf{X}^{A_1}) = X_3 \cdot (X_1 \amalg X_2) \cdot (X_4 \amalg X_5) + (1 - X_3) \cdot [(X_1 X_4) \amalg (X_2 X_5)],$$

$$\chi_2(\mathbf{X}^{A_2}) = X_8 \cdot (X_6 \amalg X_7) \cdot (X_9 \amalg X_{10}) + (1 - X_8) \cdot [(X_6 X_9) \amalg (X_7 X_{10})],$$

$$\chi_3(\mathbf{X}^{A_3}) = X_{13} \cdot (X_{11} \amalg X_{12}) \cdot (X_{14} \amalg X_{15}) + (1 - X_{13}) \cdot [(X_{11} X_{14}) \amalg (X_{12} X_{15})],$$

$$\psi = \chi_1 \cdot \chi_2 \cdot \chi_3$$

From this we get that:

$$h(\mathbf{p}) = [p_3 \cdot (p_1 \amalg p_2) \cdot (p_4 \amalg p_5) + (1 - p_3) \cdot ((p_1 p_4) \amalg (p_2 p_5))]$$
$$\cdot [p_8 \cdot (p_6 \amalg p_7) \cdot (p_9 \amalg p_{10}) + (1 - p_8) \cdot ((p_6 p_9) \amalg (p_7 p_{10}))]$$
$$\cdot [p_{13} \cdot (p_{11} \amalg p_{12}) \cdot (p_{14} \amalg p_{15}) + (1 - p_{13}) \cdot ((p_{11} p_{14}) \amalg (p_{12} p_{15}))]$$

Exercise 4.7 (cont.)

c) How many terms are there in the inclusion-exclusion formula for the reliability of this system?

SOLUTION: This system has $4 \cdot 4 \cdot 4 = 64$ minimal path sets.

The inclusion-exclusion formula contains $2^{64} - 1 \approx 1.84467 \cdot 10^{19}$ terms!

Computing say 10^9 terms pr. second, implies that calculating $h(\mathbf{p})$ would take a total of $1.84467 \cdot 10^{10}$ seconds.

$$1.84467 \cdot 10^{10} \text{ seconds } \approx 584 \text{ years}$$

Thus, i we started our computer in the year Christopher Columbus was born (1451), we would have the answer by 2035!!

Exercise 4.8

Corollary 4.3.3. If (C, ϕ) is a binary monotone system which is not coherent, then $\delta(C) = 0$.

PROOF: We recall that a *formation* of a set $A \subseteq C$ is a collection of minimal path sets whose union is equal to A. A formation is *odd* if the number of sets in the collection is *odd*, and *even* if the number of sets in the collection is *even*.

According to Theorem 4.3.2 we have for all $A \subseteq C$ that:

$$\begin{aligned}\delta(A) &= \text{The number of odd formation of } A \\ &- \text{The number of even formation of } A\end{aligned}$$

In particular $\delta(A) = 0$ if A is not a union of minimal path sets.

Exercise 4.8 (cont.)

We then recall the following result proved as part of Exercise 3.5:

Theorem

Let (C, ϕ) be a binary monotone system. Then the following three statements are equivalent:

- *$i \in C$ is relevant*
- *$i \in P$ for at least one minimal path set P*
- *$i \in K$ for at least one minimal cut set K*

If (C, ϕ) is not coherent, then there exists at least one irrelevant component $i \in C$. By the above result this implies that i is *not* contained in any minimal path set.

Since i is *not* contained in any minimal path set, the component set C is *not* a union of minimal path sets. Hence, it follows that $\delta(C) = 0$.

Exercise 4.8 (cont.)

Alternative proof: We recall that we may represent the structure function ϕ as a function defined for all subsets $B \subseteq C$ as follows:

$$\phi(B) = \phi(\mathbf{1}^B, \mathbf{0}^{C \setminus B}).$$

Thus, $\phi(B) = 1$ if B contains a minimal path set, while $\phi(B) = 0$ otherwise.

By using these two representations of the structure function, ϕ, it follows that the following three statements are equivalent:

- $i \in C$ is irrelevant.
- $\phi(1_i, \mathbf{x}) = \phi(0_i, \mathbf{x})$ for all $(\cdot_i, \mathbf{x})$.
- $\phi(B \cup i) = \phi(B)$ for all $B \subseteq (C \setminus i)$.

Exercise 4.8 (cont.)

The connection between the structure function ϕ and the signed domination fundtion δ can be expressed as follows:

$$\phi(\mathbf{x}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} x_i, \quad \text{for all } \mathbf{x} \in \{0,1\}^n$$

$$\phi(B) = \sum_{A \subseteq B} \delta(A), \quad \text{for all } B \subseteq C.$$

By using the last equation it can be shown that:

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B), \quad \text{for all } A \subseteq C.$$

Exercise 4.8 (cont.)

We then assume that $i \in C$ is irrelevant. Thus, $\phi(B \cup i) = \phi(B)$ for all $B \subseteq (C \setminus i)$. From this it follows that:

$$
\begin{aligned}
\delta(C) &= \sum_{B \subseteq C} (-1)^{|C|-|B|} \phi(B) \\
&= \sum_{B \subseteq C \setminus i} (-1)^{|C|-|B|} \phi(B) + \sum_{B \subseteq C \setminus i} (-1)^{|C|-|B \cup i|} \phi(B \cup i) \\
&= \sum_{B \subseteq C \setminus i} (-1)^{|C|-|B|} \phi(B) - \sum_{B \subseteq C \setminus i} (-1)^{|C|-|B|} \phi(B) \\
&= 0
\end{aligned}
$$