STK3405 - Exercise 5.1-5.6

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Exercise 5.1: Let (C, ϕ) be a k-out-of-n system, where $C = \{1, \ldots, n\}$. Prove that all the components of this system have the same Birnbaum measure for structural importance.

SOLUTION: Let $i \in C$ be a component in the system. We then recall that:

$$\phi(\boldsymbol{x}) = \mathrm{I}(\sum_{j \in C} x_j \geq k)$$

From this it follows that:

$$\phi(1_i, \boldsymbol{x}) = \mathrm{I}(1 + \sum_{j \in C \setminus i} x_j \geq k) = \mathrm{I}(\sum_{j \in C \setminus i} x_j \geq k - 1)$$

$$\phi(0_i, \boldsymbol{x}) = \mathrm{I}(\sum_{j \in C \setminus i} x_j \geq k)$$

Hence, we have:

$$\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x}) = \mathrm{I}(\sum_{j \in C \setminus i} x_j = k - 1)$$

We then insert this into the formula for $J_B^{(i)}$ and get:

$$J_B^{(i)} = \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} (\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x}))$$

$$= \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} \mathrm{I}(\sum_{j \in C \setminus i} x_j = k - 1)$$

The number of non-zero terms in the sum is equal to the number of ways to choose $k-1$ components out of the $n-1$ components in $C \setminus i$.

Hence, we get:

$$J_B^{(i)} = \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} \mathrm{I}(\sum_{j \in C \setminus i} x_j = k - 1) = \frac{1}{2^{n-1}} \binom{n-1}{k-1}$$

Since this holds for all $i \in C$, we conclude that all components have the same structural importance.

Exercise 5.2: Compute $J_B^{(i)}$ for the components of the bridge structure and compare their structural importance.

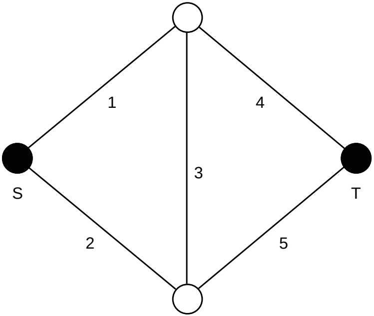
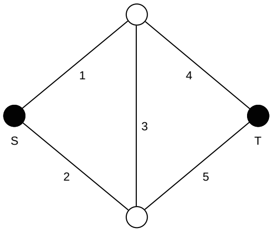


Figure: A bridge structure

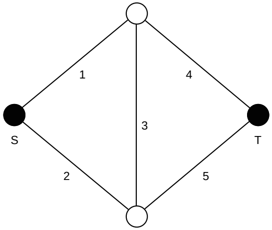


There are 2 critical path vectors for Component 3:

$$(1, 0, \cdot, 0, 1), \quad (0, 1, \cdot, 1, 0)$$

Hence, we get:

$$J_B^{(3)} = \frac{2}{2^{5-1}} = \frac{2}{16} = \frac{1}{8}$$



There are 6 critical path vectors for Component 1:

$$(\cdot, 0, 0, 1, 0), \quad (\cdot, 1, 0, 1, 0), \quad (\cdot, 0, 1, 1, 0)$$
$$(\cdot, 0, 0, 1, 1), \quad (\cdot, 0, 1, 0, 1), \quad (\cdot, 0, 1, 1, 1)$$

Hence, we get:

$$J_B^{(1)} = \frac{6}{2^{5-1}} = \frac{6}{16} = \frac{3}{8}$$

By symmetry the remaining components have the same structural importance as Component 1.

Exercise 5.3: Consider a binary monotone system(C, ϕ), where $C = \{1, \ldots, n\}$. Assume that the i'th component be in *series* with the rest of the system, while the j'th component is *not*.

Prove that:

$$J_B^{(i)} > J_B^{(j)}.$$

Since i is in series with the rest of the system, we have:

$$\phi(0_i, \boldsymbol{x}) = 0, \quad \text{for all } (\cdot_i, \boldsymbol{x}) \in \{0, 1\}^{n-1}$$

Since j is *not* in series with the rest of the system, we have:

$$\phi(0_j, \boldsymbol{x}) = 1, \quad \text{for at least one } (\cdot_j, \boldsymbol{x}) \in \{0, 1\}^{n-1}$$

Hence, we then get:

$$2^{n-1} J_B^{(i)} = \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})] = \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) + \phi(0_i, \boldsymbol{x})]$$

$$= \sum_{\boldsymbol{x}} \phi(\boldsymbol{x}) = \sum_{(\cdot_j, \boldsymbol{x})} [\phi(1_j, \boldsymbol{x}) + \phi(0_j, \boldsymbol{x})]$$

$$> \sum_{(\cdot_j, \boldsymbol{x})} [\phi(1_j, \boldsymbol{x}) - \phi(0_j, \boldsymbol{x})] = 2^{n-1} J_B^{(j)}$$

Exercise 5.4: Compute $I_B^{(i)}$ for the components of the bridge structure and compare their reliability importance.

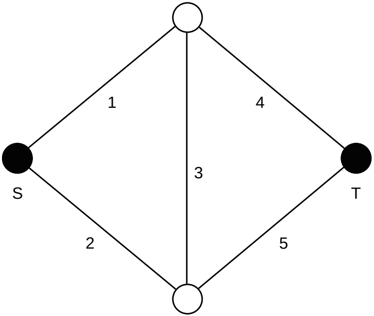


Figure: A bridge structure

SOLUTION: We know that:

$$h(\boldsymbol{p}) = p_3(p_1 \amalg p_2)(p_4 \amalg p_5) + (1 - p_3)((p_1 p_4) \amalg (p_2 p_5))$$

$$= p_3(p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5)$$

$$+ (1 - p_3)(p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5).$$

Hence, we get:

$$I_B^{(1)} = \frac{\partial h(\boldsymbol{p})}{\partial p_1} = p_3(1 - p_2)(p_4 + p_5 - p_4 p_5) + (1 - p_3)(p_4 - p_2 p_4 p_5)$$

$$I_B^{(2)} = \frac{\partial h(\boldsymbol{p})}{\partial p_2} = p_3(1 - p_1)(p_4 + p_5 - p_4 p_5) + (1 - p_3)(p_4 - p_1 p_4 p_5)$$

$$I_B^{(3)} = \frac{\partial h(\boldsymbol{p})}{\partial p_3} = (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5) - (p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5)$$

$$I_B^{(4)} = \frac{\partial h(\boldsymbol{p})}{\partial p_4} = p_3(p_1 + p_2 - p_1 p_2)(1 - p_5) + (1 - p_3)(p_1 - p_1 p_2 p_5)$$

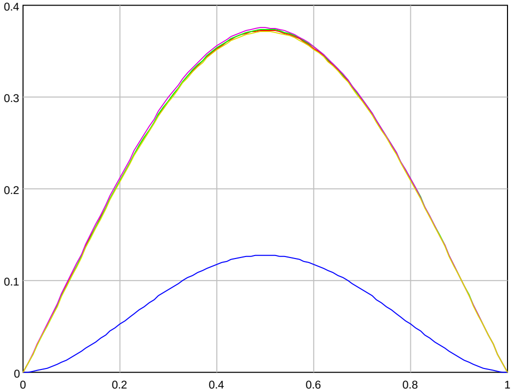
$$I_B^{(5)} = \frac{\partial h(\boldsymbol{p})}{\partial p_5} = p_3(p_1 + p_2 - p_1 p_2)(1 - p_4) + (1 - p_3)(p_1 - p_1 p_2 p_4)$$

Assuming that all components have the same reliability, that is, $p_i = p$, $i = 1, \ldots, 5$, we get:

$$\begin{aligned} I_B^{(i)} &= p(2p - p^2)(1 - p) + (1 - p)(p - p^3) \\ &= p(1 - p)(1 + 2p - 2p^2), \quad i = 1, 2, 4, 5, \end{aligned}$$

$$I_B^{(3)} = (2p - p^2)^2 - (2p^2 - p^4)$$

In order to compare the reliability importance of the components, we plot $I_B^{(i)}$ as functions of the common component reliability p:



The blue curve represents $I_B^{(3)}$, while the other curves represent $I_B^{(1)}, I_B^{(2)}, I_B^{(4)}, I_B^{(5)}$. We see that the bridge component 3 is less important than the other components for all $p \in (0, 1)$.

Assume instead that $p_1 = p_3 = p_5 = 0.9$, while $p_2 = p_4 = 0.1$. In this case we get that:

$$
\begin{aligned}
I_B^{(1)} &= p_3(1 - p_2)(p_4 + p_5 - p_4 p_5) + (1 - p_3)(p_4 - p_2 p_4 p_5) \\
&= 0.9(1 - 0.1)(0.1 + 0.9 - 0.1 \cdot 0.9) + (1 - 0.9)(0.1 - 0.1 \cdot 0.1 \cdot 0.9) \\
&= 0.9^2(1.0 - 0.09) + 0.1(0.1 - 0.009) = 0.7371 + 0.0091 = 0.7462
\end{aligned}
$$

$$
\begin{aligned}
I_B^{(2)} &= p_3(1 - p_1)(p_4 + p_5 - p_4 p_5) + (1 - p_3)(p_4 - p_1 p_4 p_5) \\
&= 0.9(1 - 0.9)(0.1 + 0.9 - 0.1 \cdot 0.9) + (1 - 0.9)(0.1 - 0.9 \cdot 0.1 \cdot 0.9) \\
&= 0.9 \cdot 0.1 \cdot (1.0 - 0.09) + 0.1 \cdot (0.1 - 0.081) = 0.0819 + 0.0019 = 0.0838
\end{aligned}
$$

$$
\begin{aligned}
I_B^{(3)} &= (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5) - (p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5) \\
&= (0.9 + 0.1 - 0.9 \cdot 0.1)^2 - (0.9 \cdot 0.1 + 0.1 \cdot 0.9 - 0.9^2 \cdot 0.1^2) \\
&= (1.0 - 0.09)^2 - (0.18 - 0.09^2) = 0.8281 - 0.1719 = 0.6562
\end{aligned}
$$

By symmetry we also get that:

$$I_B^{(4)} = I_B^{(2)} = 0.0838$$

$$I_B^{(5)} = I_B^{(1)} = 0.7462$$

Hence, with these component reliabilities we have the following ranking of importance:

$$I_B^{(1)} = I_B^{(5)} > I_B^{(3)} > I_B^{(2)} = I_B^{(4)}$$

Exercise 5.5: Assume that the component lifetimes have so-called *proportional hazards*, that is the survival function is given by:

$$\bar{F}_i(t) = 1 - F_i(t) = \exp(-\lambda_i R(t)), \quad \lambda_i > 0, t \geq 0, \quad i = 1, \ldots, n,$$

where R is a strictly increasing, differentiable function such that $R(0) = 0$, and $\lim_{t\to\infty} R(t) = \infty$. Prove that for a series structure, we have:

$$I_{B-P}^{(i)} = I_N^{(i)} = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}.$$

SOLUTION: By the definition of $\bar{F}_i(t)$ we have:

$$f_i(t) = -\frac{d\bar{F}_i(t)}{dt} = \lambda_i R'(t) \exp(-\lambda_i R(t)) = \lambda_i R'(t)\bar{F}_i(t)$$

Furthermore, since we are considering a series system, we have:

$$I_B^{(i)}(t) = \prod_{j \neq i} p_i(t) = \prod_{j \neq i} \bar{F}_j(t)$$

Then,

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt = \int_0^\infty \prod_{j \neq i} \bar{F}_j(t) \lambda_i R'(t) \bar{F}_i(t) dt$$

$$= \int_0^\infty \prod_{j=1}^n \bar{F}_j(t) \lambda_i R'(t) dt = \lambda_i \int_0^\infty e^{-\sum_{j=1}^n \lambda_j R(t)} R'(t) dt$$

$$= \lambda_i \left[-\frac{1}{\sum_{j=1}^n \lambda_j} e^{-\sum_{j=1}^n \lambda_j R(t)} \right]_{t=0}^\infty = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}$$

Here we have used that:

$$\bar{F}_i(0) = 1 \quad \Rightarrow \quad R(0) = 0,$$

$$\bar{F}_i(\infty) = 0 \quad \Rightarrow \quad R(\infty) = \infty.$$

For the Natvig measure we have:

$$E[Z_i] = \int_0^\infty \bar{F}_i(t)(-\ln \bar{F}_i(t)) I_B^{(i)}(t)dt = \int_0^\infty \prod_{j=1}^{n} \bar{F}_j(t)\lambda_i R(t)dt$$

From this it follows that:

$$I_N^{(i)} = \frac{E[Z_i]}{\sum_{k=1}^n E[Z_k]} = \frac{\lambda_i \int_0^\infty \prod \bar{F}_j(t)R(t)dt}{\sum_{k=1}^n \lambda_k \int_0^\infty \prod_{j=1}^n \bar{F}_j(t)R(t)dt} = \frac{\lambda_i}{\sum_{k=1}^n \lambda_k}$$

Thus, the Barlow Prochan and Natvig measures are the same in this case. Both rank the reliability importance of the components based on the size of the error rates λ_i. That is, the larger the error rate, the more important the component. This corresponds to the intuition that the poorest component is the most important in the series system.

Exercise 5.6: Assume that the *i*'th component is irrelevant for the system ϕ. Then, what is $I_B^{(i)}(t)$, $I_{B-P}^{(i)}$ and $I_N^{(i)}$?

SOLUTION: By pivotal decomposition we get that:

$$
\begin{aligned}
I_B^{(i)}(t) &= \frac{\partial h(\mathbf{p}(t))}{\partial p_i(t)} \\
&= h(1_i, \mathbf{p}(t)) - h(0_i, \mathbf{p}(t)) \\
&= E[\phi(1_i, \mathbf{X}(t)) - \phi(0_i, \mathbf{X}(t))] \\
&= \sum_{(\cdot_i, \mathbf{x})} (\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})) P(\mathbf{X}(t) = \mathbf{x}) \\
&= 0
\end{aligned}
$$

since the *i*'th component is irrelevant.

Since $I_B^{(i)}(t) = 0$ for all $t > 0$, it follows that:

$$I_{B-P}^{(i)} = \int_0^\infty f_i(t) I_B^{(i)}(t) dt = 0$$

$$E[Z_i] = \int_0^\infty \bar{F}_i(t)(-\ln \bar{F}_i(t)) I_B^{(i)}(t) dt = 0$$

$$I_N^{(i)} = \frac{E[Z_i]}{\sum_{j=1}^n E[Z_j]} = 0,$$

Hence, in this case we have:

$$I_{B-P}^{(i)} = I_N^{(i)} = 0.$$

Thus, according to all these measures, the reliability importance of an irrelevant component is 0 (which is intuitive).