# STK3405 – Exercise 6.3-6.6

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# Exercise 6.3

**Corollary (6.2.8)**

*Let $(C, \phi)$ be a binary monotone system of independent component states and where the component reliabilities are $p_1, \ldots, p_n$.*

*Let $P_1, \ldots, P_n$ and $K_1, \ldots, K_n$ be respectively the minimal path and cut sets of the system. Then we have:*

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\boldsymbol{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i.$$

Prove the upper bound of Corollary 6.2.8.

# Exercise 6.3 (cont.)

**SOLUTION:** We introduce the structure functions of the minimal path series structures, denoted $\rho_1(\boldsymbol{X}^{P_1}), \ldots, \rho_p(\boldsymbol{X}^{P_p})$. By Theorem 6.1.7. $\rho_1, \ldots, \rho_p$ are associated. Thus, by Theorem 6.2.1 we have that:

$$h(\boldsymbol{p}) = E[\coprod_{j=1}^{p} \rho_j(\boldsymbol{X}^{P_j})] \leq \coprod_{j=1}^{p} E[\rho_j(\boldsymbol{X}^{P_j})] = \coprod_{j=1}^{p} P(\rho_j(\boldsymbol{X}^{P_j}) = 1)$$

Furthermore, for independent component state variables, the reliabilities of the minimal path series structures are:

$$P(\rho_j(\boldsymbol{X}^{P_j}) = 1) = E[\prod_{i \in P_j} X_i] = \prod_{i \in P_j} E[X_i] = \prod_{i \in P_j} p_i$$

By combining these results we get that:

$$h(\boldsymbol{p}) \leq \coprod_{j=1}^{p} P(\rho_j(\boldsymbol{X}^{P_j}) = 1) = \coprod_{j=1}^{p} \prod_{i \in P_j} p_i$$

# Exercise 6.4

### Corollary (6.2.6)

*Consider a monotone system $(C, \phi)$, with $C = \{1, \ldots, n\}$, and with minimal path sets $P_1, \ldots, P_p$, and minimal cut sets $K_1, \ldots, K_k$.*

*Moreover, assume that the component state variables are associated, and that the component reliabilities are $p_1, \ldots, p_n$ respectively.*

*Then we have:*

$$\max_{1 \leq j \leq p} \prod_{i \in P_j} p_i \leq h \leq \min_{1 \leq j \leq k} \coprod_{i \in K_j} p_i.$$

Prove the upper bound in Corollary 6.2.6 by applying the lower bound on the dual structure function $\phi^D$.

# Exercise 6.4 (cont.)

**Solution:** Let $p_1^D, \ldots, p_n^D$ denote the reliabilities of the dual components. Then we have:

$$p_i^D = (1 - p_i), \quad i = 1, \ldots, n.$$

Similarly, we let $h^D$ denote the reliability of the dual system. Then we have:

$$h^D = 1 - h$$

Since the minimal path sets of the dual system are equal to the minimal cut sets of the original system, we can apply the lower bound on the dual system and get:

$$\max_{1 \leq j \leq k} \prod_{i \in K_j} (1 - p_i) = \max_{1 \leq j \leq k} \prod_{i \in K_j} p_i^D \leq h^D = (1 - h)$$

# Exercise 6.4 (cont.)

Rearranging the terms, we get:

$$h \leq 1 - \max_{1 \leq j \leq k} \prod_{i \in K_j} (1 - p_i) = \min_{1 \leq j \leq k} [1 - \prod_{i \in K_j} (1 - p_i)]$$

$$= \min_{1 \leq j \leq k} \coprod_{i \in K_j} p_i$$

Hence, the upper bound is proved.

# Exercise 6.5

**Corollary (6.2.8)**

*Let $(C, \phi)$ be a binary monotone system of independent component states and where the component reliabilities are $p_1, \ldots, p_n$.*

*Let $P_1, \ldots, P_n$ and $K_1, \ldots, K_n$ be respectively the minimal path and cut sets of the system. Then we have:*

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\boldsymbol{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i.$$

Prove the upper bound in Corollary 6.2.8 by applying the lower bound on the dual structure function $\phi^D$.

# Exercise 6.5 (cont.)

**Solution:** Let $p_1^D, \ldots, p_n^D$ denote the reliabilities of the dual components. Then we have:

$$p_i^D = (1 - p_i), \quad i = 1, \ldots, n.$$

Similarly, we let $h^D$ denote the reliability of the dual system. Then we have:

$$h^D = 1 - h$$

Since the minimal cut sets of the dual system are equal to the minimal path sets of the original system, we can apply the lower bound on the dual system and get:

$$\prod_{j=1}^{p} \coprod_{i \in P_j} (1 - p_i) = \prod_{j=1}^{p} \coprod_{i \in P_j} p_i^D \leq h^D = (1 - h)$$

# Exercise 6.5 (cont.)

Rearranging the terms, we get:

$$h \leq 1 - \prod_{j=1}^{p} \coprod_{i \in P_j} (1 - p_i)$$

$$= 1 - \prod_{j=1}^{p} [1 - \prod_{i \in P_j} (1 - (1 - p_i))]$$

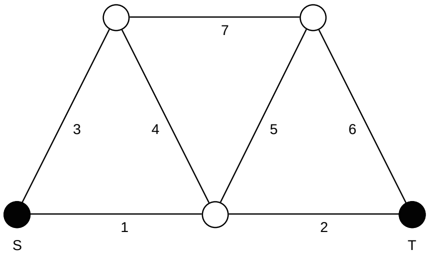
$$= 1 - \prod_{j=1}^{p} [1 - \prod_{i \in P_j} p_i] = \coprod_{j=1}^{p} \prod_{i \in P_j} p_i$$

Hence, the upper bound is proved.

# Exercise 6.6

Consider the 2-terminal undirected network system $(C, \phi)$ shown below, where $C = \{1, \ldots, 7\}$.

We also introduce the component state variables $X_1, \ldots, X_8$, and assume that these variables are independent and that $P(X_i = 1) = p$ for all $i \in C$.

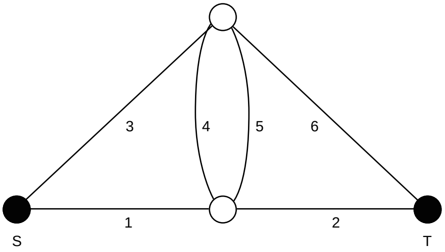



a) What is the reliability $h(p)$ of this system?

# Exercise 6.6 (cont.)

**SOLUTION:** We do a pivotal decomposition with respect to Component 7.

CASE 1. Component 7 is functioning:

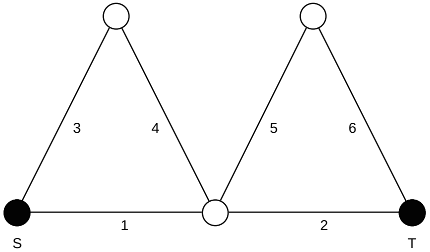



$$h(1_7, p) = (p \amalg p)(p \amalg p)^2 + [1 - (p \amalg p)](p^2 \amalg p^2)$$

$$= (p \amalg p)^3 + (1-p)^2(p^2 \amalg p^2)$$

# Exercise 6.6 (cont.)

CASE 2. Component 7 is failed:



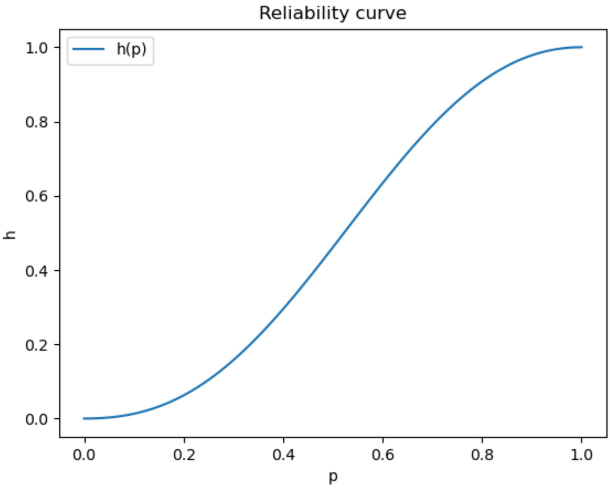

$$h(0_7, p) = (p \amalg p^2) \cdot (p \amalg p^2) = (p \amalg p^2)^2$$

# Exercise 6.6 (cont.)

The reliability of the system is then given by:

$$
\begin{aligned}
h(p) &= p \cdot h(1_7, p) + (1-p) \cdot h(0_7, p) \\
&= p[(p \amalg p)^3 + (1-p)^2(p^2 \amalg p^2)] + (1-p)(p \amalg p^2)^2
\end{aligned}
$$

# Exercise 6.6 (cont.)



# Exercise 6.6 (cont.)

### Corollary (6.2.6)

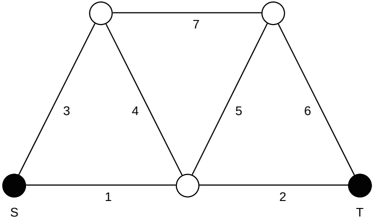
$$\max_{1\leq j\leq p} \prod_{i\in P_j} p_i \leq h \leq \min_{1\leq j\leq k} \coprod_{i\in K_j} p_i.$$

### Corollary (6.2.8)

$$\prod_{j=1}^{k} \coprod_{i\in K_j} p_i \leq h(\boldsymbol{p}) \leq \coprod_{j=1}^{p} \prod_{i\in P_j} p_i.$$

c) In the same plot, illustrate the bounds from Corollary 6.2.6 and 6.2.8. Comment on the result.

# Exercise 6.6 (cont.)




**SOLUTION:** We start out by listing the minimal path sets:

$$P_1 = \{1,2\}, \quad P_2 = \{1,5,6\}, \quad P_3 = \{2,3,4\}, \quad P_4 = \{3,4,5,6\}$$
$$P_5 = \{1,4,6,7\}, \quad P_6 = \{2,3,5,7\}, \quad P_7 = \{3,6,7\}.$$

and minimal cut sets:

$$K_1 = \{1,3\}, \quad K_2 = \{1,4,7\}, \quad K_3 = \{1,4,5,6\}$$
$$K_4 = \{2,3,4,5\}, \quad ,K_5 = \{2,5,7\}, \quad K_6 = \{2,6\}$$

# Exercise 6.6 (cont.)

**Corollary (6.2.6)**

$$\max_{1 \leq j \leq p} \prod_{i \in P_j} p_i \leq h \leq \min_{1 \leq j \leq k} \coprod_{i \in K_j} p_i.$$

This gives us the following lower and upper bounds:

$$l_1(p) = \max_{1 \leq j \leq p} \prod_{i \in P_j} p = \max\{p^2, p^3, p^4\} = p^2$$

$$u_1(p) = \min_{1 \leq j \leq k} \coprod_{i \in K_j} p_i = \min\{(p \amalg p),\ (p \amalg p \amalg p),\ (p \amalg p \amalg p \amalg p)\} = p \amalg p$$

# Exercise 6.6 (cont.)

## Corollary (6.2.8)

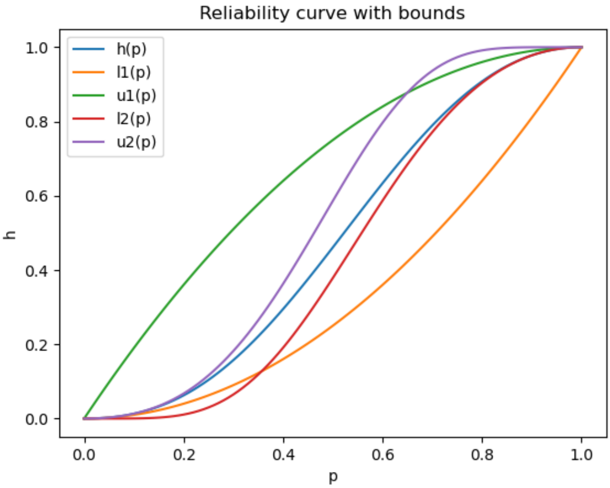
$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\boldsymbol{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i.$$

This gives us the following lower and upper bounds:

$$l_2(p) = \prod_{j=1}^{k} \coprod_{i \in K_j} p = (p \amalg p)^2 \cdot (p \amalg p \amalg p)^2 \cdot (p \amalg p \amalg p \amalg p)^2$$
$$= (1-(1-p)^2)^2 \cdot (1-(1-p)^3)^2 \cdot (1-(1-p)^4)^2$$

$$u_2(p) = \coprod_{j=1}^{p} \prod_{i \in P_j} p = p^2 \amalg p^3 \amalg p^3 \amalg p^3 \amalg p^4 \amalg p^4 \amalg p^4$$
$$= 1-(1-p^2) \cdot (1-p^3)^3 \cdot (1-p^4)^3$$

# Exercise 6.6 (cont.)



# Exercise 6.6 (cont.)

We observe that (the numbers are read off the plot):

- When $p > 0.36$ we have $l_1(p) < l_2(p)$, i.e., $l_2$ is the best lower bound.
- When $p < 0.36$ we have $l_2(p) < l_1(p)$, i.e., $l_1$ is the best lower bound.
- When $p < 0.65$ we have $u_1(p) > u_2(p)$, i.e., $u_2$ is the best upper bound.
- When $p > 0.65$ we have $u_2(p) > u_1(p)$, i.e., $u_1$ is the best lower bound.

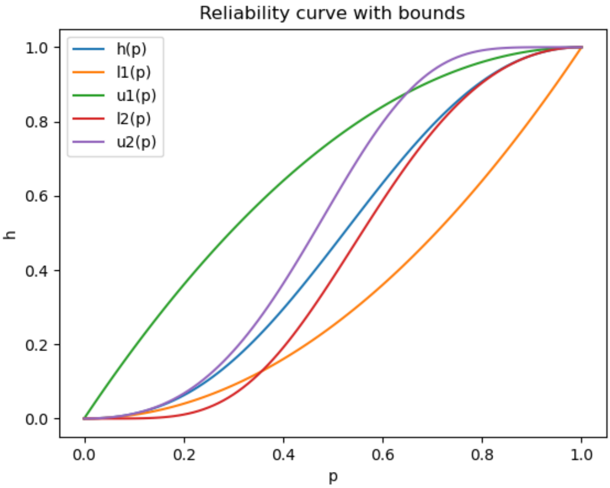
d) Is it possible to improve these bounds further?

**SOLUTION:** The bounds can be improved by always using the best of the two bounds:

$$l^*(p) = \max\{l_1(p), l_2(p)\}$$

$$u^*(p) = \min\{u_1(p), u_2(p)\}$$

# Exercise 6.6 (cont.)



# Exercise 6.6 (cont.)