STK3405 – Lecture 10

A. B. Huseby & K. R. Dahl

Department of Mathematics
University of Oslo, Norway

Chapter 8

Discrete event simulation

Section 8.1

Pure jump processes

Pure jump processes

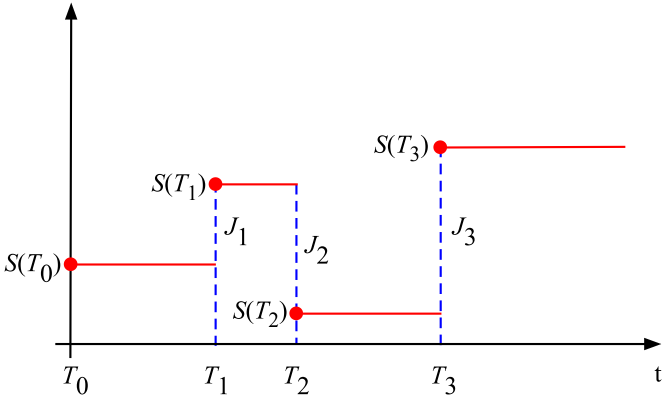


Figure: A pure jump process $S(t)$

Pure jump processes (cont.)

Let $\{S(t)\}$ be a stochastic process where $S(t)$ denotes the state of the process at time $t \geq 0$. $\{S(t)\}$ is said to be a *pure jump process* if $S(t)$ can be written as:

$$S(t) = S(0) + \sum_{j=1}^{\infty} I(T_j \leq t) J_j, \qquad t \geq 0,$$

where $0 = T_0 < T_1 < T_2 < \cdots$ is a sequence of random points of time, and $J_1, J_2, \ldots$ is a sequence of random (positive or negative) *jumps*.

In particular, for $k = 0, 1, \ldots,$ we have:

$$S(t) = S(0) + \sum_{j=1}^{k} J_j = S(T_k), \qquad \text{for all } t \in [T_k, T_{k+1}).$$

From this it follows that the state function $S(t)$ is *piecewise constant* and *right-continuous* in t, with jumps at $T_1 < T_2 < \cdots$.

In order to keep track of how the process evolves only the *event points* need to be considered.

Regular pure jump processes

Let $\{S(t)\}$ be pure jump process, and let:

$$\begin{aligned} N(t) &= \sum_{j=1}^{\infty} I(T_j \leq t) \\ &= \text{The number of jumps in [0, t].} \end{aligned}$$

We say that $\{S(t)\}$ is *regular* if $P(N(t) < \infty) = 1$ for all $t > 0$.

NOTE

$$\begin{aligned} P(N(t) < \infty) &= P(\lim_{k\to\infty} T_k = \infty) \\ &= P(\lim_{k\to\infty} \sum_{j=1}^{k} \Delta_j = \infty), \end{aligned}$$

where $\Delta_j = T_j - T_{j-1}, j = 1, 2, \ldots$.

Regular pure jump processes (cont.)

Proposition (8.1.1)

Let $\{S(t)\}$ *be a pure jump process with jumps at:*

$$T_1 < T_2 < \cdots$$

Moreover, we let $T_0 = 0$ *and introduce the non-negative random variables* $\Delta_j = T_j - T_{j-1}$, $j = 1, 2, \ldots$.

If the sequence $\{\Delta_j\}$ *contains an infinite subsequence* $\{\Delta_{k_j}\}$ *of independent, identically distributed random variables such that* $E[\Delta_{k_j}] = d > 0$, *then* $\{S(t)\}$ *is regular.*

Regular pure jump processes (cont.)

PROOF: By the strong law of large numbers it follows that:

$$P(\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \Delta_{k_j} = d) = 1.$$

This implies that the series $\sum_{j=1}^{\infty} \Delta_{k_j}$ is divergent with probability one. Hence, since obviously $\sum_{j=1}^{\infty} \Delta_{k_j} \leq \sum_{j=1}^{\infty} \Delta_j$, the result follows.

Regular pure jump processes (cont.)

Proposition (8.1.2)

Let $\{S(t)\}$ be a regular pure jump process with jumps at $T_1 < T_2 < \cdots$. Then $\lim_{t \to s^-} S(t)$ exists for every $s > 0$ with probability one.

PROOF: Let $0 \leq t < s < \infty$, and consider the set:

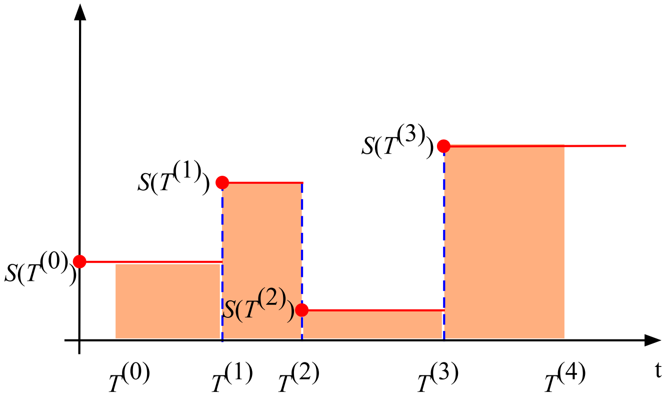
$$\mathcal{T} = \{T_j : t \leq T_j < s\} \cup \{t\}.$$

Since S is regular, the set $\mathcal{T}$ is finite with probability one. Moreover, $\mathcal{T}$ is non-empty since $t \in \mathcal{T}$. Thus, this set contains a maximal element, which we denote by t'. Moreover, since every element in $\mathcal{T}$ is less than s, then so is t'.

From this it follows that the interval (t', s) is nonempty.

At the same time (t', s) does not contain any jumps, so $S(t)$ is constant throughout this interval. Hence, $\lim_{t \to s^-} S(t)$ exists. Since s was arbitrary chosen, this holds for any $s > 0$.

Regular pure jump processes (cont.)



If $u = T^{(0)}$ and $v = T^{(k+1)}$, then:

$$\int_u^v S(t)dt = \sum_{j=0}^{k} S(T^{(j)})(T^{(j+1)} - T^{(j)}).$$

Regular pure jump processes (cont.)

Proposition (8.1.3)

Let $\{S(t)\}$ *be a regular pure jump process with jumps at* $T_1 < T_2 < \cdots$, *and let* $0 \leq u < v < \infty$.

Assume that $\{T_j : u < T_j < v\} = \{T^{(1)}, \ldots, T^{(k)}\}$, *where* $T^{(1)} < \cdots < T^{(k)}$.

Moreover, we define $T^{(0)} = u$ *and* $T^{(k+1)} = v$.

Then we have:

$$\int_u^v S(t)dt = \sum_{j=0}^{k} S(T^{(j)})(T^{(j+1)} - T^{(j)}).$$

Regular pure jump processes (cont.)

NOTE: Since $\{S(t)\}$ is regular, the number of elements in the set $\{T_j : u < T_j < v\}$ is finite with probability one.

Thus, this set can almost surely be written in the form $\{T^{(1)}, \ldots, T^{(k)}\}$, for some suitable $k < \infty$.

Since S is right-continuous and piecewise constant, it follows that $S(t) = S(T^{(j)})$ for all $t \in [T^{(j)}, T^{(j+1)})$, $j = 0, 1, \ldots, k$.

Thus, we have:

$$\int_{T^{(j)}}^{T^{(j+1)}} S(t)dt = S(T^{(j)})(T^{(j+1)} - T^{(j)}), \quad j = 0, 1, \ldots, k.$$

The result then follows by adding up the contributions to the integral from each of the $k + 1$ intervals $[T^{(0)}, T^{(1)}), \ldots, [T^{(k)}, T^{(k+1)})$

Regular pure jump processes (cont.)

Proposition (8.1.4)

Let $\{S_1(t)\}, \ldots, \{S_n(t)\}$ *be n regular pure jump processes, and let* $H(t) = H(\boldsymbol{S}(t))$, *where* $\boldsymbol{S}(t) = (S_1(t) \ldots, S_n(t))$, $t \geq 0$.

Then $\{H(t)\}$ *is a regular pure jump process as well.*

That is, $H(t) = H(\boldsymbol{S}(t))$ *is:*

- *Piecewise constant*
- *Right-continuous in* t,
- *The number of jumps in* $[0, t]$ *is finite with probability one for all* $t > 0$

Regular pure jump processes (cont.)

PROOF: Let $\mathcal{T}_i$ be the set of jump points of $\{S_i(t)\}$, $i = 1, \ldots, n$, and let $\mathcal{T}$ be the set of jump points of the process $\{H(t)\}$.

Since the state value of H cannot change unless there is a change in the state value of at least one of the elementary processes, it follows that $\mathcal{T} \subseteq (\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n)$.

Thus, $H(t)$ is piecewise constant and right-continuous in t. Moreover, for any finite interval $[0, t]$ we also have:

$$\mathcal{T} \cap [0, t] \subseteq [(\mathcal{T}_1 \cap [0, t]) \cup \cdots \cup (\mathcal{T}_n \cap [0, t])].$$

By regularity $(\mathcal{T}_i \cap [0, t])$ is finite almost surely for $i = 1, \ldots, n$. Hence, $\mathcal{T} \cap [0, t]$ is finite almost surely as well, implying that $\{H(t)\}$ is regular.

Binary monotone systems of repairable components

Binary monotone systems of repairable components

Consider (C, ϕ), a binary monotone system of *n repairable* components.

Component state processes: $\{X_1(t)\}, \ldots, \{X_n(t)\}$, where:

$$X_i(t) = \text{the state of component } i \text{ at time } t \geq 0, \quad i \in C.$$

Component state vector: $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$

System state process: $\{\phi(t)\}$, where:

$$\phi(t) = \phi(\boldsymbol{X}(t)) = \text{the state of the system at time } t \geq 0$$

Repairable components

We introduce the following random variables:

- U_{ij} = The jth lifetime of the ith component
- D_{ij} = The jth repair time of the ith component

For $i = 1, \ldots, n$:

- $U_{i1}, U_{i2}, \ldots$ are i.i.d. with mean value $0 < \mu_i < \infty$
- $D_{i1}, D_{i2}, \ldots$ are i.i.d with mean value $0 < \nu_i < \infty$

All lifetimes and repair times are assumed to be *independent*. Thus, in particular the component processes $\{X_1(t)\}, \ldots, \{X_n(t)\}$ are independent of each other.

NOTE: The U_{ij}'s and the D_{ij}'s are waiting times between state changes for the components.

Repairable components

Now we let for $i = 1, \ldots, n$:

$$\begin{aligned} T_{i,1} &= U_{i1}, \\ T_{i,2} &= U_{i1} + D_{i1} \\ T_{i,3} &= U_{i1} + D_{i1} + U_{i2}, \\ T_{i,4} &= U_{i1} + D_{i1} + U_{i2} + D_{i2} \\ &\cdots \end{aligned}$$

Moreover, let $J_j = (-1)^j$. We may then write:

$$X_i(t) = X(0) + \sum_{j=1}^{\infty} I(T_{ij} \leq t) J_j, \quad i = 1, \ldots, n.$$

By Proposition 8.1.1 $\{X_1(t)\}, \ldots, \{X_n(t)\}$ are regular pure jump processes.

By Proposition 8.1.4 $\{\phi(t)\}$ is a regular pure jump process as well.

Availability

Let $A_i(t)$ be the availability of the ith component at time t. That is, for $i = 1, \ldots, n$ we have:

$$A_i(t) = \Pr(X_i(t) = 1) = E[X_i(t)].$$

By renewal theory the corresponding stationary availabilities are given by:

$$A_i = \lim_{t \to \infty} A_i(t) = \frac{\mu_i}{\mu_i + \nu_i}, \qquad i = 1, \ldots, n.$$

Introduce $\boldsymbol{A}(t) = (A_1(t), \ldots, A_n(t))$ and $\boldsymbol{A} = (A_1, \ldots, A_n)$. The system availability at time t is given by:

$$A_\phi(t) = \Pr(\phi(\boldsymbol{X}(t)) = 1) = E[\phi(\boldsymbol{X}(t))] = h(\boldsymbol{A}(t)),$$

where h is the system's reliability function. The corresponding stationary availability is given by:

$$A_\phi = \lim_{t \to \infty} A_\phi(t) = h(\boldsymbol{A})$$

Criticality

The component i is said to be *critical* at time t if

$$\psi_i(\boldsymbol{X}(t)) = \phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t)) = 1.$$

$\psi_i(\boldsymbol{X}(t))$ is the *criticality state* of component i at time t.

The Birnbaum measure of importance of component i at time t, $I_B^{(i)}(t)$, is the probability that i is critical at time t:

$$\begin{aligned} I_B^{(i)}(t) &= \Pr(\psi_i(\boldsymbol{X}(t)) = 1) = E[\psi_i(\boldsymbol{X}(t))] \\ &= h(1_i, \boldsymbol{A}(t)) - h(0_i, \boldsymbol{A}(t)). \end{aligned}$$

The corresponding stationary measure is given by:

$$I_B^{(i)} = \lim_{t \to \infty} I_B^{(i)}(t) = h(1_i, \boldsymbol{A}) - h(0_i, \boldsymbol{A}).$$

Simulating repairable systems

Event model

Let (C, ϕ) be a binary monotone system with component state processes: $\{X_1(t)\}, \ldots, \{X_n(t)\}$.

- $E_{i1}, E_{i2}, \ldots$ are the events affecting the process $\{X_i(t)\}$
- $T_{i1}, T_{i2}, \ldots$ are the corresponding points of time for these events

Assuming that all lifetimes and repair times have *absolutely continuous distributions*, all the events happen at *distinct* points of time almost surely, i.e., all the T_{ij}s are distinct numbers.

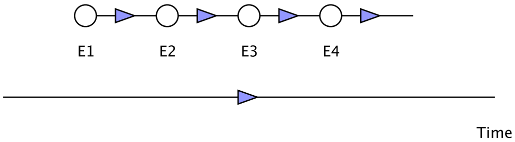
We assume that the events are sorted with respect to their respective points of time, so that $T_{i1} < T_{i2} < \cdots$.

Event model (cont.)

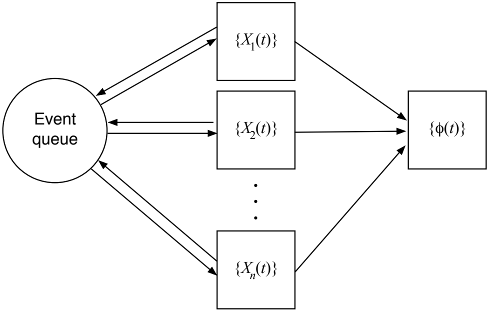
At the system level the event set is the *union* of all the component event sets.

- Let $E^{(1)}, E^{(2)}, \ldots$ denote the system events sorted with respect to their respective points of time
- Let $T^{(1)} < T^{(2)} < \cdots$ be the corresponding points of time

Each system event corresponds to a unique component event, organised in a dynamic queue sorted with respect to the points of time of the events:



Program flow



- The components post initial events to the event queue
- The event queue processes events in chronological order, and notifies the components when the events occur. As soon as an event is processed, it is removed from the queue.
- The component updates its state, posts a new event to the queue, and notifies the system about the state change

Sampling events

Although the system state and component states stay constant between events, it is of interest to sample the state values at *predefined* points of time. Thus, we introduce yet another type of event, called a *sampling events* spread out evenly on the timeline.

- Let $e_1, e_2, \ldots$ denote the sampling events
- Let $t_1 < t_2 < \cdots$ are the corresponding points of time

Typically $t_j = j \cdot \Delta$ for some suitable $\Delta > 0$, $j = 1, 2, \ldots$.

The sampling events are placed into the queue in the same way as for the ordinary events.

Estimating availability and importance

Pointwise estimates of availability and importance

Goal: Estimate $A_\phi(t)$ and $I_B^{(1)}(t), \ldots, I_B^{(n)}(t)$ for $0 \leq t \leq t_N$

Solution: Pointwise estimates of $A_\phi(t)$ and $I_B^{(1)}(t), \ldots, I_B^{(n)}(t)$ for $t \in \{t_1, \ldots, t_N\}$, and use interpolation between these points.

In each simulation we sample the values of ϕ and $\psi_1, \ldots, \psi_n$ at each sampling point $t_1, \ldots, t_N$. We denote the sth simulated result of the component state vector process by $\{\boldsymbol{X}_s(t)\}$, $s = 1, \ldots, M$, and obtain the following estimates for $j = 1, \ldots, N$:

$$\hat{A}_\phi(t_j) = \frac{1}{M} \sum_{s=1}^{M} \phi(\boldsymbol{X}_s(t_j)),$$

$$\hat{I}_B^{(i)}(t_j) = \frac{1}{M} \sum_{s=1}^{M} \psi_i(\boldsymbol{X}_s(t_j)).$$

Interval estimates of availability and importance

Alternative idea: Use average simulated availability and criticalities from each interval $(t_{j-1}, t_j]$, $j = 1, \ldots, N$ as estimates for the availability and criticalities at the midpoints of these intervals.

We then obtain the following estimates for $j = 1, \ldots, N$:

$$\tilde{A}_\phi(\bar{t}_j) = \frac{1}{M} \sum_{s=1}^{M} \frac{1}{\Delta} \sum_{k \in \mathcal{E}_{sj}} \phi(\boldsymbol{X}_s(T_s^{(k)}))(T_s^{(k+1)} - T_s^{(k)}),$$

$$\tilde{I}_B^{(i)}(\bar{t}_j) = \frac{1}{M} \sum_{s=1}^{M} \frac{1}{\Delta} \sum_{k \in \mathcal{E}_{sj}} \psi_i(\boldsymbol{X}_s(T_s^{(k)}))(T_s^{(k+1)} - T_s^{(k)}),$$

where $\mathcal{E}_{sj}$ denotes the index set of the events in $(t_{j-1}, t_j]$ in the sth simulation, and $\bar{t}_j = (t_{j-1} + t_j)/2$.

Interval estimates of availability and importance

- The integral formula given in Proposition 8.1.3 implies that $\tilde{A}_\phi(\bar{t}_j)$ and $\tilde{I}_B^{(i)}(\bar{t}_j)$ are unbiased and strongly consistent estimates of the corresponding average availability and criticality in the intervals $[t_{j-1}, t_j)$ respectively.
- By choosing Δ so that the availabilities and criticalities are relatively stable within each interval, the interval estimates are approximately unbiased estimates for $A_\phi(\bar{t}_j)$ and $I_B^{(i)}(\bar{t}_j)$ as well.
- The resulting interval estimates stabilize much faster than the pointwise estimates.
- Interpolation is used to estimate $A_\phi(t)$ and $I_B^{(i)}(t)$ between the interval midpoints.
- Since all process information is used in the estimates, satisfactory curve estimates can be obtained for a much higher value of Δ than the one needed for the pointwise estimates.

Estimates of asymptotic availability and importance

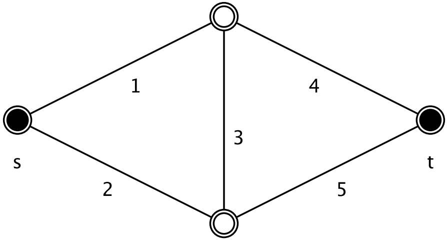
We may also obtain estimates of the asymptotic availability and importance by calculating averages over the intervals $(0, t_j]$, $j = 1, 2, \ldots$:

$$\bar{A}_\phi(t_j) = \frac{1}{M} \sum_{s=1}^{M} \frac{1}{t_j} \sum_{k \in \mathcal{F}_{sj}} \phi(\boldsymbol{X}_s(T_s^{(k)}))(T_s^{(k+1)} - T_s^{(k)}),$$

$$\bar{I}_B^{(i)}(t_j) = \frac{1}{M} \sum_{s=1}^{M} \frac{1}{t_j} \sum_{k \in \mathcal{F}_{sj}} \psi_i(\boldsymbol{X}_s(T_s^{(k)}))(T_s^{(k+1)} - T_s^{(k)}),$$

where $\mathcal{F}_{sj}$ denotes the index set of the events in $(0, t_j]$ in the sth simulation.

Example: A bridge system



The five components in the system have exponential lifetime and repair time distributions with mean values 1 time unit.

Objective: Estimate $A_\phi(t)$ and $I_B^{(1)}(t), \ldots, I_B^{(5)}(t)$ for $t \in [0, 1000]$.

Stationary values

Since in this *very particular case* all the lifetimes and repair times are exponentially distributed with the *same mean*, component availabilities can easily be calculated analytically:

$$N_i(t) = \text{Number of failure/repair events affecting comp. } i \text{ in } [0, t].$$

Now, we note that:

- $N_i(t)$ has a Poisson distribution with mean t
- $X_i(t) = 1$ if and only if $N_i(t)$ is even

Hence:

$$A_i(t) = \sum_{k=0}^{\infty} \Pr(N_i(t) = 2k) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} e^{-t}.$$

Stationary values (cont.)

Convergence of the system availability:

$$|A_\phi(t) - A_\phi| < 10^{-15}, \text{ for } t > 20$$

Convergence of the Birnbaum measures of importance:

$$|I_B^{(i)}(t) - I_B^{(i)}| < 10^{-15}, \text{ for } t > 20, \; i = 1, \ldots, 5.$$

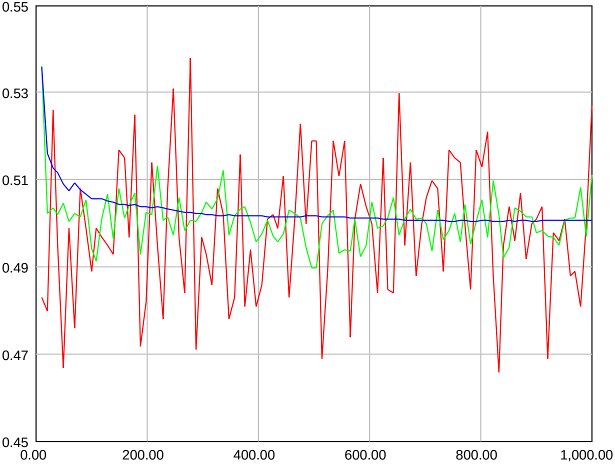


Figure: Availability curve estimates, $\hat{A}_\phi(t)$ (red curve), $\tilde{A}_\phi(t)$ (green curve) and $\bar{A}_\phi(t)$ (blue curve), $M = 1000$ simulations, $N = 100$ sample points, $\Delta = 10$ units

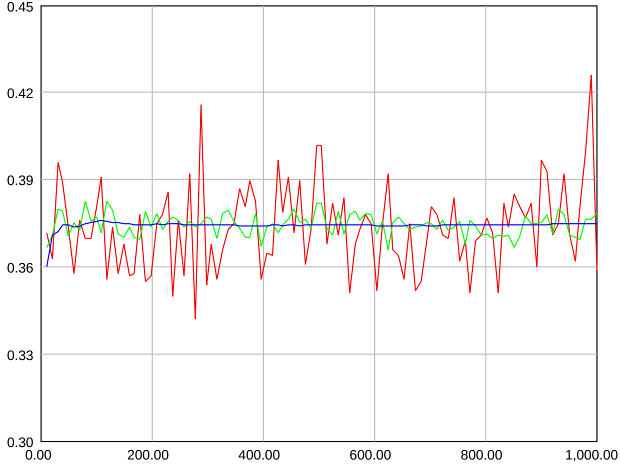


Figure: Importance curve estimates, $\hat{I}_B^{(1)}(t)$ (red curve) and $\tilde{I}_B^{(1)}(t)$ (green curve) and $\bar{I}_B^{(1)}(t)$ (blue curve), $M = 1000$ simulations, $N = 100$ sample points, $\Delta = 10$ units