STK3405 – Lecture 11

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The Barlow-Proschan measure of reliability importance

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Definition (Barlow-Proschan measure)

Let (C, ϕ) be a non-trivial binary monotone system where $C = \{1, \ldots, n\}$. Moreover, let T_i denote the lifetime of component i, $i \in C$, and let S denote the lifetime of the system.

The Barlow-Proschan measure of the reliability importance of component $i \in C$ is defined as:

$$\begin{aligned} I_{B-P}^{(i)} &= P(\text{Component } i \text{ fails at the same time as the system}) \\ &= P(T_i = S). \end{aligned}$$

The Barlow-Proschan measure of reliability importance (cont.)

Theorem (Probability of system failure)

Let (C, ϕ) be a non-trivial binary monotone system where $C = \{1, \ldots, n\}$. Moreover, let T_i denote the lifetime of component i, $i \in C$, and let S denote the lifetime of the system.

Assume that $T_1, \ldots, T_n$ are independent and absolutely continuously distributed.

Then S is absolutely continuously distributed as well, and we have:

$$\sum_{i=1}^{n} I_{B-P}^{(i)} = 1.$$

The Barlow-Proschan measure of reliability importance (cont.)

Theorem (Integral formula for the Barlow-Proschan measure)

Let (C, ϕ) *be a non-trivial binary monotone system where* $C = \{1, \ldots, n\}$*, and let* T_i *denote the lifetime of component* i*,* $i \in C$*.*

Assume that $T_1, \ldots, T_n$ *are independent, absolutely continuously distributed with densities* $f_1, \ldots, f_n$ *respectively. Then, we have:*

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt,$$

where $I_B^{(i)}(t)$ *denotes the Birnbaum measure of the reliability importance of component* i *at time* t*.*

The Natvig measure of reliability importance

The Natvig measure of reliability importance

- **The Barlow-Proschan measure:** Components which have long lifetimes compared to the system lifetime, are the most important components.
- **The Natvig measure:** Components which greatly reduce the remaining system lifetime by failing, are the most important components.

The Natvig measure of reliability importance (cont.)

Definition (The Natvig measure)

Let (C, ϕ) be a non-trivial binary monotone system where $C = \{1, \ldots, n\}$. Moreover, for $i \in C$ let:

$$Z_i = \text{Reduction of remaining lifetime for the system due to } i \text{ failing.}$$

The *Natvig measure for the reliability importance of component* i, denoted $I_N^{(i)}$, is defined by:

$$I_N^{(i)} = \frac{E[Z_i]}{\sum_{j=1}^{n} E[Z_j]}$$

where we assume that $E[Z_i]$ is finite.

The Natvig measure of reliability importance (cont.)

It is easy to show that $0 \leq I_N^{(i)} \leq 1$ for all $i \in C$, and that $\sum_{i=1}^{n} I_N^{(i)} = 1$.

We also have the following theorem:

Theorem (Integral formula for the Natvig measure)

Let (C, ϕ) be a binary monotone system where $C = \{1, \ldots, n\}$, and where the components are independent and their lifetimes, $T_1, \ldots, T_n$ are absolutely continuously distributed. Then we have:

$$E[Z_i] = \int_0^\infty \bar{F}_i(t)(-\ln(\bar{F}_i(t))) I_B^{(i)}(t) dt, \quad i \in C,$$

where $\bar{F}_i(t) = P(T_i > t)$ for all $i \in C$.

The Natvig measure of reliability importance (cont.)

Example: Assume that $f_i(t) = \lambda_i e^{-\lambda_i t}$ for $i \in C$. Then for all $i \in C$ we have:

$$\bar{F}_i(t) = \int_t^\infty f_i(u)du = e^{-\lambda_i t}$$

Hence, we get that:

$$\bar{F}_i(t)(-\ln(\bar{F}_i(t))) = \lambda_i t \cdot e^{-\lambda_i t} = t \cdot f_i(t)$$

Thus, in this case we have:

$$I_N^{(i)} \propto E[Z_i] = \int_0^\infty I_B^{(i)}(t) t \cdot f_i(t) dt, \quad i \in C$$

At the same time:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt.$$

The Natvig measure of reliability importance (cont.)

Conclusion: When the component lifetimes are independent and exponentially distributed, the Natvig measure puts more weight on later points of time than early points of time compared to the Barlow-Proschan measure.

Chapter 6

Association and bounds for the system reliability

Associated random variables

Associated random variables

Definition (Associated random variables)

Let $T_1, \ldots, T_n$ be random variables, and let $\boldsymbol{T} = (T_1, \ldots, T_n)$. We say that $T_1, \ldots, T_n$ are associated if

$$\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0,$$

for all binary non-decreasing functions Γ and Δ.

NOTE: We only require $\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0$ for all *binary* non-decreasing functions.

Associated random variables (cont.)

Theorem (Generalized covariance property)

Let $T_1, \ldots, T_n$ be associated random variables, and f and g functions which are non-decreasing in each argument such that $\mathrm{Cov}(f(\boldsymbol{T}), g(\boldsymbol{T}))$ exists, i.e.,

$$E[|f(\boldsymbol{T})|] < \infty, E[|g(\boldsymbol{T})|] < \infty, E[|f(\boldsymbol{T})g(\boldsymbol{T})|] < \infty.$$

Then we have:

$$\mathrm{Cov}(f(\boldsymbol{T}), g(\boldsymbol{T})) \geq 0.$$

Associated random variables (cont.)

Theorem (Properties of Associated variables)

Associated random variables have the following properties:

(i) *Any subset of a set of associated random variables also consists of associated random variables.*

(ii) *A single random variable is always associated.*

(iii) *Non-decreasing functions of associated random variables are associated.*

(iv) *If two sets of associated random variables are independent, then their union is a set of associated random variables.*

Associated random variables (cont.)

We note that (*i*) follows from (*iii*). However, we can also prove this property directly:

Let $T_1, \ldots, T_n$ be a set of associated random variables, and let $A \subset \{1, \ldots, n\}$. We would like to prove that $\{T_i\}_{i \in A}$ is a set of associated random variables. To do so, let Γ_A, Δ_A be arbitrary, binary functions which are non-decreasing in all of their arguments T_i, $i \in A$. We then define:

$$\Gamma(\boldsymbol{T}) = \Gamma_A(\boldsymbol{T}^A), \quad \Delta(\boldsymbol{T}) = \Delta_A(\boldsymbol{T}^A).$$

From this it follows that:

$$\mathrm{Cov}(\Gamma_A(\boldsymbol{T}^A), \Delta_A(\boldsymbol{T}^A)) = \mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0,$$

where the inequality follows from the definition because we have assumed that $T_1, \ldots, T_n$ is a set of associated random variables. Hence, (*i*) is proved.

Associated random variables (cont.)

To prove (*ii*) we let T be a random variable, and let Γ, Δ be arbitrary, binary functions which are non-decreasing in T. Then, since Γ, Δ are binary and non-decreasing in T, there are only two possible cases:

CASE 1. $\Gamma(T) \leq \Delta(T)$ for all T,

CASE 2. $\Gamma(T) \geq \Delta(T)$ for all T.

We consider Case 1 only, as Case 2 can be handled similarly. Then, we have:

$$\begin{aligned}
\mathrm{Cov}(\Gamma(T), \Delta(T)) &= E[\Gamma(T)\Delta(T)] - E[\Gamma(T)]E[\Delta(T)] \\
&= E[\Gamma(T)] - E[\Gamma(T)]E[\Delta(T)] \\
&= E[\Gamma(T)](1 - E[\Delta(T)]) \geq 0,
\end{aligned}$$

where the second equality follows from the fact that we are in case 1. The last inequality holds because $\Gamma(T), \Delta(T) \in \{0, 1\}$ for all T. Hence, (*ii*) is proved as well.

Associated random variables (cont.)

To prove (*iii*) we let $T_1, \dots, T_n$ be associated, and let $\boldsymbol{T} = (T_1, \dots, T_n)$. Moreover, we let $S_i = f_i(\boldsymbol{T})$, $i = 1, \dots, m$, where $f_1, \dots, f_m$ are non-decreasing functions, and let $\boldsymbol{S} = (S_1, \dots, S_m)$.

Finally, let $\Gamma = \Gamma(\boldsymbol{S})$ and $\Delta = \Delta(\boldsymbol{S})$ be binary non-decreasing functions. Then $\Gamma(\boldsymbol{S}) = \Gamma(f_1(\boldsymbol{T}), \dots, f_m(\boldsymbol{T}))$ and $\Delta(\boldsymbol{S}) = \Delta(f_1(\boldsymbol{T}), \dots, f_m(\boldsymbol{T}))$ are non-decreasing functions of $\boldsymbol{T}$ as well.

Hence, by the definition it follows that:

$$\mathrm{Cov}(\Gamma(\boldsymbol{S}), \Delta(\boldsymbol{S})) = \mathrm{Cov}(\Gamma(f_1(\boldsymbol{T}), \dots, f_m(\boldsymbol{T})), \Delta(f_1(\boldsymbol{T}), \dots, f_m(\boldsymbol{T}))) \geq 0.$$

Hence, we conclude that $S_1, \dots, S_m$ are associated as well.

Associated random variables (cont.)

To prove (*iv*) we let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two vectors of associated random variables, and assume that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent of each other. Moreover, we assume that $\Gamma = \Gamma(\boldsymbol{X}, \boldsymbol{Y})$ and $\Delta = \Delta(\boldsymbol{X}, \boldsymbol{Y})$ are binary and non-decreasing functions in both $\boldsymbol{X}$ and $\boldsymbol{Y}$. Then we have:

$$\mathrm{Cov}(\Gamma, \Delta) = E[\mathrm{Cov}(\Gamma, \Delta|\boldsymbol{X})] + \mathrm{Cov}[E(\Gamma|\boldsymbol{X}), E(\Delta|\boldsymbol{X})].$$

We then note that for any $\boldsymbol{x}$, $\Gamma(\boldsymbol{x}, \boldsymbol{Y})$ and $\Delta(\boldsymbol{x}, \boldsymbol{Y})$ are binary non-decreasing functions of $\boldsymbol{Y}$. Hence, we must have:

$$\mathrm{Cov}(\Gamma(\boldsymbol{X}, \boldsymbol{Y}), \Delta(\boldsymbol{X}, \boldsymbol{Y})|\boldsymbol{X} = \boldsymbol{x}) = \mathrm{Cov}(\Gamma(\boldsymbol{x}, \boldsymbol{Y}), \Delta(\boldsymbol{x}, \boldsymbol{Y})) \geq 0, \text{ for all } \boldsymbol{x},$$

where the equality follows since $\boldsymbol{Y}$ is independent of $\boldsymbol{X}$, while the inequality follows since $\boldsymbol{Y}$ is associated.

Associated random variables (cont.)

This implies that:

$$E[\mathrm{Cov}(\Gamma, \Delta | \boldsymbol{X})] \geq 0.$$

Moreover, $E[\Gamma(\boldsymbol{x}, \boldsymbol{Y})]$ and $E[\Delta(\boldsymbol{x}, \boldsymbol{Y})]$ are non-decreasing (but not necessarily binary) functions of $\boldsymbol{x}$. Hence, since $\boldsymbol{X}$ is associated, it follows by previous results that:

$$\mathrm{Cov}[E(\Gamma | \boldsymbol{X}), E(\Delta | \boldsymbol{X})] \geq 0.$$

Note that since Γ and Δ are binary, we must have that $E(\Gamma | \boldsymbol{X}) \in [0, 1]$ and $E(\Delta | \boldsymbol{X}) \in [0, 1]$ with probability one. Thus, obviously $\mathrm{Cov}[E(\Gamma | \boldsymbol{X}), E(\Delta | \boldsymbol{X})]$ exists.

Combining these results implies that:

$$\mathrm{Cov}(\Gamma, \Delta) \geq 0.$$

Thus, we conclude that $(\boldsymbol{X}, \boldsymbol{Y})$ is associated.

Associated random variables (cont.)

Theorem (Independence)

Let $T_1, \ldots, T_n$ be independent. Then, they are also associated.

PROOF: (Induction on n.) The result obviously holds for $n = 1$ by property (*ii*).

Assume that the theorem holds for $n = m - 1$. That is, $\{T_1, \ldots, T_{m-1}\}$ is a set of associated random variables.

Moreover, by property (*ii*), $\{T_m\}$ is associated as well.

By the assumption, these two sets are independent. Hence, it follows from property (*iv*) that their union $\{T_1, \ldots, T_{m-1}, T_m\}$ is a set of associated random variables.

Thus, the result is proved by induction.

Associated random variables (cont.)

Theorem (Absolute dependence)

Let $T_1, \ldots, T_n$ be completely positively dependent random variables, i.e.,

$$P(T_1 = T_2 = \ldots = T_n) = 1.$$

Then they are associated.

PROOF: Let Γ, Δ be binary functions which are non-decreasing in each argument and let $\boldsymbol{T} = (T_1, \ldots, T_n)$ and $\boldsymbol{T}_1 = (T_1, \ldots, T_1)$. By the assumption it follows that $\boldsymbol{T}$ and $\boldsymbol{T}_1$ must have the same distribution. Hence, we get that:

$$\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) = \mathrm{Cov}(\Gamma(\boldsymbol{T}_1), \Delta(\boldsymbol{T}_1)) \geq 0$$

where the final inequality follows by property (*ii*) and the definition.

Associated random variables (cont.)

Theorem (Association of paths and cuts)

Let $X_1, \ldots, X_n$ be the associated or independent component state variables of a monotone system (C, ϕ). Moreover, let the minimal path series structures of the system be $(P_1, \rho_1), \ldots, (P_p, \rho_p)$, where:

$$\rho_j(\boldsymbol{X}^{P_j}) = \prod_{i \in P_j} X_i, \quad j = 1, \ldots, p.$$

Then, $\rho_1, \ldots, \rho_p$ are associated.

Similarly, let the minimal cut parallel structures of the system be $(K_1, \kappa_1), \ldots, (K_k, \kappa_k)$, where:

$$\kappa_j(\boldsymbol{X}^{K_j}) = \coprod_{i \in K_j} X_i, \quad j = 1, \ldots, k.$$

Then, $\kappa_1, \ldots, \kappa_k$ are associated.

Associated random variables (cont.)

Theorem (Extension of property (iii))

Let $\boldsymbol{T} = (T_1, \ldots, T_n)$ be associated, and let:

$$U_i = g_i(\boldsymbol{T}), \quad i = 1, \ldots, m,$$

where g_i, $i = 1, \ldots, m$ are non-increasing functions. Then, $\boldsymbol{U} = (U_1, \ldots, U_m)$ is associated.

Associated random variables (cont.)

PROOF: Let Γ, Δ be binary non-decreasing functions, and introduce $\boldsymbol{U} = \boldsymbol{g}(\boldsymbol{T}) = (g_1(\boldsymbol{T}), \ldots, g_m(\boldsymbol{T}))$. Then let:

$$\bar{\Gamma}(\boldsymbol{T}) = 1 - \Gamma(\boldsymbol{g}(\boldsymbol{T})) = 1 - \Gamma(\boldsymbol{U})$$

$$\bar{\Delta}(\boldsymbol{T}) = 1 - \Delta(\boldsymbol{g}(\boldsymbol{T})) = 1 - \Delta(\boldsymbol{U})$$

It follows that $\bar{\Gamma}$ and $\bar{\Delta}$ are binary and non-decreasing in T_i, $i = 1, \ldots, n$. Since $\boldsymbol{T}$ is associated, it follows that:

$$\begin{aligned}
\mathrm{Cov}(\Gamma(\boldsymbol{U}), \Delta(\boldsymbol{U}))) &= \mathrm{Cov}(1 - \bar{\Gamma}(\boldsymbol{T}), 1 - \bar{\Delta}(\boldsymbol{T})) \\
&= \mathrm{Cov}(1, 1) + \mathrm{Cov}(1, -\bar{\Delta}(\boldsymbol{T})) + \mathrm{Cov}(-\bar{\Gamma}(\boldsymbol{T}), 1) \\
&+ \mathrm{Cov}(-\bar{\Gamma}(\boldsymbol{T}), -\bar{\Delta}(\boldsymbol{T})) \\
&= \mathrm{Cov}(\bar{\Gamma}(\boldsymbol{T}), \bar{\Delta}(\boldsymbol{T})) \geq 0.
\end{aligned}$$

Hence, we conclude that $\boldsymbol{U}$ is associated.

Associated random variables (cont.)

Theorem (Bivariate association)

Let X and Y be two binary random variables. Then, X and Y are associated if and only if

$$\mathrm{Cov}(X, Y) \geq 0.$$

PROOF: Assume first that X and Y are associated. We may then choose $\Gamma(X, Y) = X$ and $\Delta(X, Y) = Y$.

Since obviously Γ and Δ are binary and non-decreasing functions, it follows by definition that $\mathrm{Cov}(X, Y) = \mathrm{Cov}(\Gamma, \Delta) \geq 0$.

Associated random variables (cont.)

Assume conversely that $\text{Cov}(X, Y) \geq 0$. We want to prove that this implies that $\text{Cov}(\Gamma(X, Y), \Delta(X, Y)) \geq 0$ for all binary non-decreasing functions, Γ and Δ.

The only choices for Γ and Δ are :

$$\Gamma_1 \equiv 0, \quad \Gamma_2 = X \cdot Y, \quad \Gamma_3 = X, \quad \Gamma_4 = Y, \quad \Gamma_5 = X \amalg Y, \quad \Gamma_6 \equiv 1.$$

These functions can be ordered as follows:

$$\Gamma_1 \leq \Gamma_2 \leq \left\{ \begin{array}{c} \Gamma_3 \\ \Gamma_4 \end{array} \right\} \leq \Gamma_5 \leq \Gamma_6.$$

Associated random variables (cont.)

Assume first that Γ and Δ from the set $\{\Gamma_1, \ldots, \Gamma_6\}$ such that $\Gamma(X, Y) \leq \Delta(X, Y)$. We then have:

$$\begin{aligned} \mathrm{Cov}(\Gamma, \Delta) &= E(\Gamma \cdot \Delta) - E(\Gamma) \cdot E(\Delta) \\ &= E(\Gamma) - E(\Gamma) \cdot E(\Delta) = E(\Gamma)[1 - E(\Delta)] \geq 0. \end{aligned}$$

The only possibility left is $\Gamma = \Gamma_3 = X$ and $\Delta = \Gamma_4 = Y$. However, in this case we get that:

$$\mathrm{Cov}(\Gamma, \Delta) = \mathrm{Cov}(X, Y) \geq 0,$$

where the last inequality follows by the assumption. Hence, we conclude that $\mathrm{Cov}(\Gamma, \Delta) \geq 0$ for all binary non-decreasing functions, and thus the result is proved.

Upper and lower bounds for the reliability of monotone systems

Associated random variables

Definition (Associated random variables)

Let $T_1, \ldots, T_n$ be random variables, and let $\boldsymbol{T} = (T_1, \ldots, T_n)$. We say that $T_1, \ldots, T_n$ are associated if

$$\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0,$$

for all binary non-decreasing functions Γ and Δ.

NOTE: We only require $\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0$ for all *binary* non-decreasing functions.

Associated random variables (cont.)

Theorem (Generalized covariance property)

Let $T_1, \ldots, T_n$ be associated random variables, and f and g functions which are non-decreasing in each argument such that $\mathrm{Cov}(f(\boldsymbol{T}), g(\boldsymbol{T}))$ exists, i.e.,

$$E[|f(\boldsymbol{T})|] < \infty, E[|g(\boldsymbol{T})|] < \infty, E[|f(\boldsymbol{T})g(\boldsymbol{T})|] < \infty.$$

Then we have:

$$\mathrm{Cov}(f(\boldsymbol{T}), g(\boldsymbol{T})) \geq 0.$$

Bounds for the system reliability

Theorem (6.2.1)

Let $T_1, \ldots, T_n$ be associated random variables such that $0 \leq T_i \leq 1$, $i = 1, \ldots, n$. We then have:

$$E[\prod_{i=1}^{n} T_i] \geq \prod_{i=1}^{n} E[T_i]$$

$$E[\coprod_{i=1}^{n} T_i] \leq \coprod_{i=1}^{n} E[T_i]$$

Bounds for the system reliability (cont.)

PROOF: Note that since $0 \leq T_i \leq 1$, both T_i and $S_i = 1 - T_i$ are non-negative random variables, $i = 1, \ldots, n$. Hence, the product functions $\prod_{i=j}^n T_i$ and $\prod_{i=j}^n S_i$ are non-decreasing in each argument, $j = 1, \ldots, n$.

By using the generalized covariance property, we find:

$$E[\prod_{i=1}^n T_i] - E[T_1]E[\prod_{i=2}^n T_i] = \mathrm{Cov}(T_1, \prod_{i=2}^n T_i) \geq 0,$$

since the product function is non-decreasing in each argument.

This implies that:

$$E[\prod_{i=1}^n T_i] \geq E[T_1]E[\prod_{i=2}^n T_i].$$

By repeated use of this inequality, we get the first inequality.

Bounds for the system reliability (cont.)

From the extension of property (iii), $S_1, \ldots, S_n$ are associated random variables. Moreover, $0 \leq S_i \leq 1$, $i = 1, \ldots, n$, so we can apply the first inequality to these variables.

From this it follows that:

$$\begin{aligned} E[\coprod_{i=1}^{n} T_i] &= 1 - E[\prod_{i=1}^{n}(1 - T_i)] = 1 - E[\prod_{i=1}^{n} S_i] \\ &\leq 1 - \prod_{i=1}^{n} E(S_i) = 1 - \prod_{i=1}^{n}(1 - E[T_i]) \\ &= \coprod_{i=1}^{n} E[T_i], \end{aligned}$$

so the second inequality is proved as well.

Bounds for the system reliability (cont.)

We apply the theorem to the component state variables $X_1, \ldots, X_n$:

- The first inequality says that for a series structure of associated components, an incorrect assumption of independence will lead to an *underestimation* of the system reliability.
- The second inequality says that for a parallel structure, an incorrect assumption of independence between the components will lead to an *overestimation* of the system reliability.

Since most systems are not purely series or purely parallel, we conclude that for an arbitrary structure, we *cannot say for certain what the consequences of an incorrect assumption of independence will be*.

Fortunately, it is still possible to obtain bounds on the system reliability.