STK3405 – Lecture 12

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Associated random variables

Definition (Associated random variables)

Let $T_1, \ldots, T_n$ be random variables, and let $\boldsymbol{T} = (T_1, \ldots, T_n)$. We say that $T_1, \ldots, T_n$ are associated if

$$\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0,$$

for all binary non-decreasing functions Γ and Δ.

NOTE: We only require $\mathrm{Cov}(\Gamma(\boldsymbol{T}), \Delta(\boldsymbol{T})) \geq 0$ for all *binary* non-decreasing functions.

Associated random variables (cont.)

Theorem (Properties of Associated variables)

Associated random variables have the following properties:

(i) *Any subset of a set of associated random variables also consists of associated random variables.*

(ii) *A single random variable is always associated.*

(iii) *Non-decreasing functions of associated random variables are associated.*

(iv) *If two sets of associated random variables are independent, then their union is a set of associated random variables.*

Associated random variables (cont.)

Theorem (Association of paths and cuts)

Let $X_1, \ldots, X_n$ be the associated or independent component state variables of a monotone system (C, ϕ). Moreover, let the minimal path series structures of the system be $(P_1, \rho_1), \ldots, (P_p, \rho_p)$, where:

$$\rho_j(\boldsymbol{X}^{P_j}) = \prod_{i \in P_j} X_i, \quad j = 1, \ldots, p.$$

Then, $\rho_1, \ldots, \rho_p$ are associated.

Similarly, let the minimal cut parallel structures of the system be $(K_1, \kappa_1), \ldots, (K_k, \kappa_k)$, where:

$$\kappa_j(\boldsymbol{X}^{K_j}) = \coprod_{i \in K_j} X_i, \quad j = 1, \ldots, k.$$

Then, $\kappa_1, \ldots, \kappa_k$ are associated.

Bounds for the system reliability

Theorem (6.2.1)

Let $T_1, \ldots, T_n$ be associated random variables such that $0 \leq T_i \leq 1$, $i = 1, \ldots, n$. We then have:

$$E[\prod_{i=1}^{n} T_i] \geq \prod_{i=1}^{n} E[T_i]$$

$$E[\coprod_{i=1}^{n} T_i] \leq \coprod_{i=1}^{n} E[T_i]$$

Bounds for the system reliability (cont.)

In the next results we interpret T_i, $i = 1, \ldots, n$ as the lifetimes of components in a binary monotone system.

Theorem (6.2.2)

Let $T_1, \ldots, T_n$ be non-negative associated random variables. Then, for all t_i, $i = 1, \ldots, n$:

$$P[\bigcap_{i=1}^{n}(T_i > t_i)] \geq \prod_{i=1}^{n} P[T_i > t_i],$$

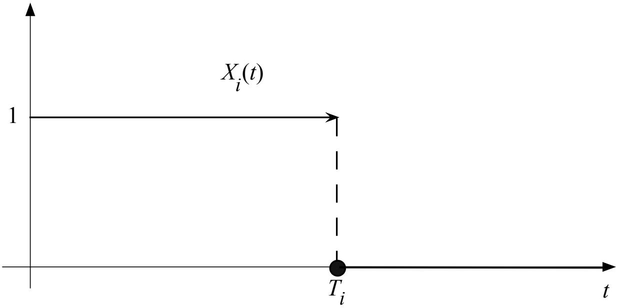
$$P[\bigcap_{i=1}^{n}(T_i \leq t_i)] \geq \prod_{i=1}^{n} P[T_i \leq t_i].$$

Bounds for the system reliability (cont.)

PROOF: Interpreting T_i as the lifetime of the ith component, we may introduce the binary component state process $\{X_i(t)\}$, $i = 1, \ldots, n$, where:

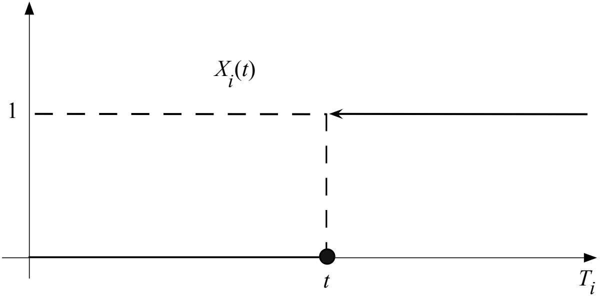
$$X_i(t) = \begin{cases} 1, & t < T_i \\ 0, & t \geq T_i. \end{cases}$$

Bounds for the system reliability (cont.)



In this we have plotted $X_i(t)$ as a function of $t \geq 0$ for a given value of T_i. The plot shows that $X_i(t)$ is a non-increasing function of t.

Bounds for the system reliability (cont.)



Alternatively, $X_i(t)$ can be plotted as a function of the lifetime T_i for a *fixed point of time* t. We see that $X_i(t)$ is a non-decreasing function of T_i.

Hence, since $T_1, \ldots, T_n$ are assumed to be associated random variables, it follows that $X_1(t), \ldots, X_n(t)$ are associated for all $t \geq 0$.

Bounds for the system reliability (cont.)

Applying the first part of Theorem 6.2.1, we get:

$$
\begin{aligned}
P[\bigcap_{i=1}^{n}(T_i > t_i)] &= P[\bigcap_{i=1}^{n}(X_i(t_i) = 1)] \\
&= P[\prod_{i=1}^{n} X_i(t_i) = 1] = E[\prod_{i=1}^{n} X_i(t_i)] \\
&\geq \prod_{i=1}^{n} E[X_i(t_i)] = \prod_{i=1}^{n} P[X_i(t_i) = 1] \\
&= \prod_{i=1}^{n} P[T_i > t_i],
\end{aligned}
$$

and thus the first inequality is proved.

Bounds for the system reliability (cont.)

Applying the second part of Theorem 6.2.1, we get:

$$\begin{aligned}
P[\bigcap_{i=1}^{n}(T_i \leq t_i)] &= P[\bigcap_{i=1}^{n}(X_i(t_i) = 0)] = 1 - P[\bigcup_{i=1}^{n}(X_i(t_i) = 1)] \\
&= 1 - P[\coprod_{i=1}^{n} X_i(t_i) = 1] = 1 - E[\coprod_{i=1}^{n} X_i(t_i)] \\
&\geq 1 - \coprod_{i=1}^{n} E[X_i(t_i)] = 1 - [1 - \prod_{i=1}^{n}(1 - E[X_i(t_i)])] \\
&= \prod_{i=1}^{n}(1 - P[X_i(t_i) = 1]) = \prod_{i=1}^{n} P[T_i \leq t_i],
\end{aligned}$$

and thus the second inequality is proved.

Bounds for the system reliability (cont.)

Corollary (6.2.3)

Let $T_1, \ldots, T_n$ be non-negative associated random variables. Then:

$$P(\min_{1 \leq i \leq n} T_i > t) = P(\bigcap_{i=1}^{n} T_i > t) \geq \prod_{i=1}^{n} P(T_i > t)$$

$$P(\max_{1 \leq i \leq n} T_i > t) = 1 - P(\bigcap_{i=1}^{n} T_i \leq t) \leq \prod_{i=1}^{n} P(T_i > t).$$

NOTE: If (C, ϕ) is a series system with component lifetimes $T_1, \ldots, T_n$, then the system lifetime is equal to $\min_{1 \leq i \leq n} T_i$.

If (C, ϕ) is a parallel system with component lifetimes $T_1, \ldots, T_n$, then the system lifetime is equal to $\max_{1 \leq i \leq n} T_i$.

Bounds for the system reliability (cont.)

We skip the following results:

- Theorem 6.2.4 (Very crude upper and lower bounds)
- Theorem 6.2.5 (We incorporate this result in Corollary 6.2.6)
- Theorem 6.2.7 (We incorporate this result in Corollary 6.2.8)

Bounds for the system reliability (cont.)

Corollary (6.2.6)

Consider a monotone system (C, ϕ), where $C = \{1, \ldots, n\}$ and with minimal path sets $P_1, \ldots, P_p$ and minimal cut sets $K_1, \ldots, K_k$.

Moreover, assume that the component state variables, $X_1, \ldots, X_n$ are associated with component reliabilities $p_1, \ldots, p_n$. Then we have:

$$\max_{1 \leq j \leq p} \prod_{i \in P_j} p_i \leq h \leq \min_{1 \leq j \leq k} \coprod_{i \in K_j} p_i.$$

Bounds for the system reliability (cont.)

PROOF: We have that:

$$\min_{i \in P_r} X_i \leq \max_{1 \leq r \leq p} \min_{i \in P_r} X_i = \phi(\boldsymbol{X}) = \min_{1 \leq s \leq k} \max_{i \in K_s} X_i \leq \max_{i \in K_s} X_i,$$

for all $r = 1, \ldots, p$ and all $s = 1, \ldots, k$.

This implies that:

$$P(\min_{i \in P_r} X_i = 1) \leq h \leq P(\max_{i \in K_s} X_i = 1)$$

for all $r = 1, \ldots, p$ and all $s = 1, \ldots, k$.

Hence, we must have:

$$\max_{1 \leq j \leq p} P[\min_{i \in P_j} X_i = 1] \leq h \leq \min_{1 \leq j \leq k} P[\max_{i \in K_j} X_i = 1].$$

Bounds for the system reliability (cont.)

Furthermore, since $X_1, \ldots, X_n$ are associated, we may use Theorem 6.2.1 and get:

$$P[\min_{i \in P_j} X_i = 1] = E[\prod_{i \in P_j} X_i] \geq \prod_{i \in P_j} E[X_i] = \prod_{i \in P_j} p_i$$

$$P[\max_{i \in K_j} X_i = 1] = E[\coprod_{i \in K_j} X_i] \leq \coprod_{i \in K_j} E[X_i] = \coprod_{i \in K_j} p_i$$

Inserting these inequalities into the bounds on the previous slide we get:

$$\max_{1 \leq j \leq p} \prod_{i \in P_j} p_i \leq h \leq \min_{1 \leq j \leq k} \coprod_{i \in K_j} p_i.$$

Bounds for the system reliability (cont.)

Corollary (6.2.8)

Let (C, ϕ) be a binary monotone system where $C = \{1, \ldots, n\}$, and assume that the component state variables, $X_1, \ldots, X_n$ are independent with component reliabilities $p_1, \ldots, p_n$.

Moreover, let $P_1, \ldots, P_p$ and $K_1, \ldots, K_k$ be respectively the minimal path and cut sets of the system.

Then we have:

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\boldsymbol{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i.$$

Bounds for the system reliability (cont.)

PROOF: We introduce:

$$\rho_j(\boldsymbol{X}) = \prod_{i \in P_j} X_i, \quad j = 1, \ldots, p,$$

$$\kappa_j(\boldsymbol{X}) = \coprod_{i \in K_j} X_i, \quad j = 1, \ldots, k.$$

Since $\rho_1, \ldots, \rho_p$ and $\kappa_1, \ldots, \kappa_k$ are non-decreasing functions of $\boldsymbol{X}$, they are associated. Hence, by Theorem 6.2.1 we have:

$$h(\boldsymbol{p}) = E[\coprod_{j=1}^{p} \prod_{i \in P_j} X_i] = E[\coprod_{j=1}^{p} \rho_j(\boldsymbol{X})] \leq \coprod_{j=1}^{p} E[\rho_j(\boldsymbol{X})]$$

$$h(\boldsymbol{p}) = E[\prod_{j=1}^{k} \coprod_{i \in K_j} X_i] = E[\prod_{j=1}^{k} \kappa_j(\boldsymbol{X})] \geq \prod_{j=1}^{k} E[\kappa_j(\boldsymbol{X})]$$

Bounds for the system reliability (cont.)

Moreover, since the component state variables, $X_1, \ldots, X_n$, are *independent*, we have:

$$E[\rho_j(\boldsymbol{X})] = E[\prod_{i \in P_j} X_i] = \prod_{i \in P_j} p_i,$$

$$E[\kappa_j(\boldsymbol{X})] = E[\coprod_{i \in K_j} X_i] = \coprod_{i \in K_j} p_i.$$

Inserting this into the bounds on the previous slide, i.e.:

$$\prod_{j=1}^{k} E[\kappa_j(\boldsymbol{X})] \leq h(\boldsymbol{p}) \leq \coprod_{j=1}^{p} E[\rho_j(\boldsymbol{X})],$$

we get:

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\boldsymbol{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i.$$

Bounds for the system reliability (cont.)

In the coming examples we shall compare the bounds from Corollary 6.2.6 to those from Corollary 6.2.8.

Let $h_n(p)$ denote the reliability of a parallel system of n components where all components have the same reliability p. We then have:

$$h_2(p) = p \amalg p = 1 - (1-p)(1-p)$$
$$= 1 - (1 - 2p + p^2) = 2p - p^2,$$

$$h_3(p) = p \amalg p \amalg p = 1 - (1-p)(1-p)(1-p)$$
$$= 1 - (1 - 3p + 3p^2 - p^3) = 3p - 3p^2 + p^3.$$

Bounds for the system reliability (cont.)

EXAMPLE 1: A 3-out-of-4 system with $p_i = p$, $i = 1, 2, 3, 4$ where all the component state variables are independent.

The minimal path sets for the 3-out-of-4 system are:

$$P_1 = \{1, 2, 3\},\ P_2 = \{1, 2, 4\},\ P_3 = \{1, 3, 4\},\ P_4 = \{2, 3, 4\},$$

and the minimal cut sets are:

$$K_1 = \{1, 2\},\ K_2 = \{1, 3\},\ K_3 = \{1, 4\},\ K_4 = \{2, 3\},\ K_5 = \{2, 4\},\ K_6 = \{3, 4\}.$$

Bounds for the system reliability (cont.)

The lower and upper bounds in Corollary 6.2.6 are denoted by $l_1(p)$ and $u_1(p)$ respectively, and are given by:

$$l_1(p) = \max_{1 \leq j \leq 4} \prod_{i \in P_j} p = \max_{1 \leq j \leq 4} p^3 = p^3,$$

$$u_1(p) = \min_{1 \leq j \leq 6} \coprod_{i \in K_j} p = \min_{1 \leq j \leq 6} h_2(p) = 2p - p^2.$$

The lower and upper bounds in Corollary 6.2.8 are denoted by $l_2(p)$ and $u_2(p)$ respectively, and are given by:

$$l_2(p) = \prod_{j=1}^{6} \coprod_{i \in K_j} p = \prod_{j=1}^{6} h_2(p) = (2p - p^2)^6,$$

$$u_2(p) = \coprod_{j=1}^{4} \prod_{i \in P_j} p = h_2(p^3) \amalg h_2(p^3) = 2(2p^3 - p^6) - (2p^3 - p^6)^2.$$

Bounds for the system reliability (cont.)

The *true* reliability of the 3-out-of-4 system is given by:

$$h(p) = \sum_{i=3}^{4} \binom{4}{i} p^i (1-p)^{n-i} = 4p^3(1-p) + p^4.$$

Bounds for the system reliability (cont.)

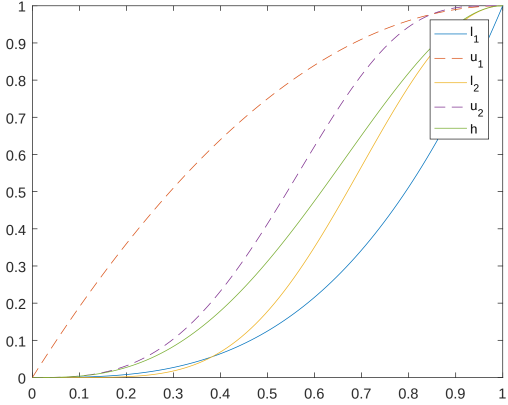


Figure: The true reliability function h as well as the bounds l_1, u_1, l_2, u_2.

Bounds for the system reliability (cont.)

EXAMPLE 2: A bridge system with $p_i = p$, $i = 1, 2, 3, 4, 5$ where all the component state variables are independent.

The minimal path sets for the bridge system are:

$$P_1 = \{1, 4\},\ P_2 = \{1, 3, 5\},\ P_3 = \{2, 3, 4\},\ P_4 = \{2, 5\},$$

and the minimal cut sets are:

$$K_1 = \{1, 2\},\ K_2 = \{1, 3, 5\},\ K_3 = \{2, 3, 4\},\ K_4 = \{4, 5\}.$$

Bounds for the system reliability (cont.)

The lower and upper bounds in Corollary 6.2.6 are denoted by $l_1(p)$ and $u_1(p)$ respectively, and are given by:

$$l_1(p) = \max_{1 \leq j \leq 4} \prod_{i \in P_j} p = \max(p^2, p^3, p^3, p^2) = p^2,$$

$$u_1(p) = \min_{1 \leq j \leq 4} \coprod_{i \in K_j} p = \min(h_2(p), h_3(p), h_3(p), h_2(p)) = 2p - p^2.$$

The lower and upper bounds in Corollary 6.2.8 are denoted by $l_2(p)$ and $u_2(p)$ respectively, and are given by:

$$l_2(p) = \prod_{j=1}^{4} \coprod_{i \in K_j} p = h_2(p)^2 \cdot h_3(p)^2,$$

$$u_2(p) = \coprod_{j=1}^{4} \prod_{i \in P_j} p = h_2(p^2) \amalg h_2(p^3).$$

Bounds for the system reliability (cont.)

The *true* reliability of the bridge system is given by:

$$h(p) = p \cdot h_2(p)^2 + (1 - p) \cdot h_2(p^2).$$

Bounds for the system reliability (cont.)

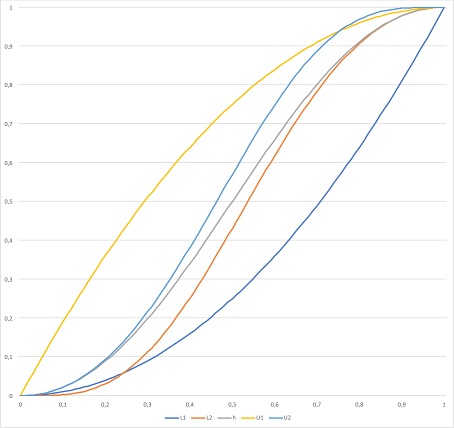


Figure: The true reliability function h as well as the bounds l_1, u_1, l_2, u_2.

Bounds for the system reliability (cont.)

We see that in both examples the bounds from Corollary 6.2.8 are better than those from Corollary 6.2.6 for *most* of the p-values.

NOTE:

- The lower bound l_1 from Corollary 6.2.6 is better than l_2 from Corollary 6.2.8 for small values of p.
- The upper bound u_1 from Corollary 6.2.6 is better than u_2 from Corollary 6.2.8 for large p-values.

In order to always get the best bounds, we may introduce l^* and u^* defined as follows:

$$l^* = \max(l_1, l_2),$$

$$u^* = \min(u_1, u_2)$$