STK3405 – Lecture 14

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Case study - Reliability analysis of a network for transmission of electronic pulses

Multistate systems

We consider a system with component set $C = \{1, \ldots, n\}$ and binary component state processes $\{X_i(t)\}_{i \in C}$, where:

$$X_i(t) = \mathrm{I}(\text{Component } i \text{ is functioning at time } t \geq 0), \quad i \in C.$$

We also introduce $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$.

At any given point of time t the system state is a function of the components states, and we introduce the *structure function*:

$$\phi(t) = \phi(\boldsymbol{X}(t)) = \text{The state of the system at time } t.$$

In this case, however, the system is a *multistate system*. Thus, the set of system states, denoted $\mathcal{S}$, may contain more than two states.

Our main task is to find the distribution of $\phi(t)$ for any $t \geq 0$.

The structure function

The distribution of $\phi(t)$:
In principle we can compute the distribution of $\phi(t)$ by total state space enumeration:

$$P(\phi(t) = s) = \sum_{\boldsymbol{x}} \mathrm{I}(\phi(\boldsymbol{x}) = s) \cdot P(\boldsymbol{X}(t) = \boldsymbol{x}), \quad s \in \mathcal{S}.$$

However, since the number of terms in this sum grows exponentially in n, this often becomes too time consuming.

State space reduction:
The computational complexity can be reduced considerably if we can find new processes $\{Y_1(\boldsymbol{X}(t))\}, \ldots, \{Y_m(\boldsymbol{X}(t))\}$ such that $m << n$, and such that:

$$\phi(t) = \phi(\boldsymbol{Y}(\boldsymbol{X}(t))),$$

where $\boldsymbol{Y}(\boldsymbol{X}(t)) = (Y_1(\boldsymbol{X}(t)), \ldots, Y_m(\boldsymbol{X}(t)))$.

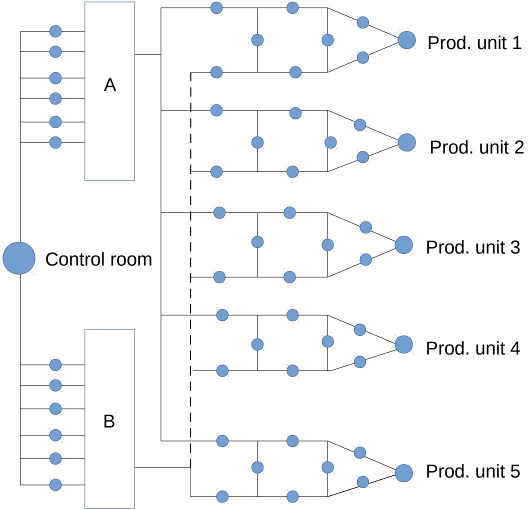
State space reduction (cont.)

Assuming that the joint distribution of $\boldsymbol{Y}(\boldsymbol{X}(t))$ is easily determined, we can then find the distribution of $\phi(t)$ by using the following formula:

$$P(\phi(t) = s) = \sum_{\boldsymbol{y}} \mathrm{I}(\phi(\boldsymbol{y}) = s) \cdot P(\boldsymbol{Y}(\boldsymbol{X}(t)) = \boldsymbol{y}), \quad s \in \mathcal{S}.$$

In general, the number of terms in this sum grows exponentially in m. However, if $m << n$, the number of terms may be reduced considerably.

A network for transmission of electronic pulses



A network for transmission of electronic pulses (cont.)

The purpose of the system shown in the previous slide is to ensure communication between a control room and 5 production units.

The system consists of:

- 6 wires beween the control room and the connection unit A
- 6 wires beween the control room and the connection unit B
- 5 identical subsystems, one for each production unit, transporting electronic pulses from the connection units, A and B, to the production unit. Each subsystem contains 8 components.

In total we have $n = 6 + 6 + 5 \cdot 8 = 52$ components.

The number of production units which can be controlled by a connection unit is limited by the number of functioning wires between the control room and the respective connection unit.

The structure of the subsystems

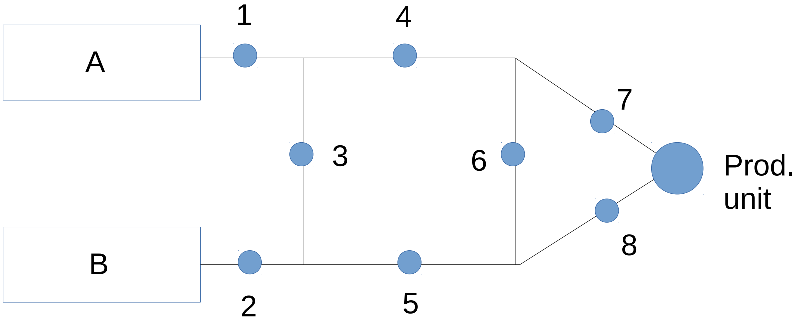


Figure: A subsystem transporting electronic pulses from the connection units, *A* and *B*, to a production unit.

Assumptions

- All the 52 the components have idependent and exponentially distributed lifetimes
- The 12 wires have the same failure rate r_0.
- The 5 subsystems have identical stochastic properties.
- The failure rates of the 8 components of a subsystem are $r_1, \ldots, r_8$ respectively.

For a given point of time $t \geq 0$ we introduce the survival probabilities:

$$q_i(t) = \exp(-r_i t), \quad i = 0, 1, \ldots, 8.$$

NOTE: A survival probability at time t is the probability that the component is functioning at time t, or equivalently the reliability of the component at time t.

The structure function

We define the state of the system at time $t \geq 0$ as:

$$\phi(t) = \text{The number of production units which can be controlled at time } t$$

Thus, it follows that the set of possible system states is:

$$\mathcal{S} = \{0, 1, \dots, 5\}$$

Since the total number of components is 52, a total state space enumeration would consist of a sum of $2^{52} \approx 4.504 \cdot 10^{15}$ terms.

This is way too many terms to handle. Thus, we must look for some simplifications.

Intermediate variables

In order to simplify the calculations we introduce intermediate variables:

$Y_1(t) =$ The number of functioning wires at the connection unit A at time t

$Y_2(t) =$ The number of functioning wires at the connection unit B at time t

$Y_3(t) =$ The number of prod.units connected to both A and B at time t

$Y_4(t) =$ The number of prod.units connected to A only at time t

$Y_5(t) =$ The number of prod.units connected to B only at time t

The state of the system can now be expressed as:

$$\phi(t) = W_1(t) + W_2(t) + W_3(t),$$

where:

$$W_1(t) = \min\{Y_1(t), Y_4(t)\}, \quad W_2(t) = \min\{Y_2(t), Y_5(t)\},$$
$$W_3(t) = \min\{(Y_1(t) - W_1(t)) + (Y_2(t) - W_2(t)), Y_3(t)\}.$$

Intermediate variables (cont.)

We observe that:

- $W_1(t)$ is the number of production units which can only be controlled through connection unit A.
- $W_2(t)$ is the number of production units which can only be controlled through connection unit B.
- $(Y_1(t) - W_1(t))$ is the left-over capacity at connection unit A.
- $(Y_2(t) - W_2(t))$ is the left-over capacity at connection unit B.
- $W_3(t)$ is the number of production units which can be handled by the total left-over capacity at connection unit A and connection unit B.

Note that we may have zero left-over capacity at one or both connection units.

The distribution of $Y_1(t)$ and $Y_2(t)$

We observe that $Y_1(t)$ and $Y_2(t)$ are sums of 6 independent, identically distributed binary random variables.

Since the 12 wires have a survival probability of $q_0(t) = \exp(-r_0 t)$, it follows that:

$$P(Y_i(t) = y) = \binom{6}{y}[q_0(t)]^y[1 - q_0(t)]^{6-y}, \quad y = 0, 1, \ldots, 6, \quad i = 1, 2.$$

The distribution of $Y_3(t)$, $Y_4(t)$ and $Y_5(t)$

We start by observing that each of the 5 subsystems can be in 4 different states:

$$S_{AB} = \text{The prod. unit can communicate with both connection units}$$
$$S_{A} = \text{The prod. unit can communicate with connection unit } A$$
$$S_{B} = \text{The prod. unit can communicate with connection unit } B$$
$$S_{\emptyset} = \text{The prod. unit cannot communicate with any connection unit}$$

We recall that all subsystems have identical stochastic properties. Thus, we may introduce:

$$q_{AB}(t) = \text{The probability that a prod.unit is in state } S_{AB} \text{ at time } t$$
$$q_{A}(t) = \text{The probability that a prod.unit is in state } S_{A} \text{ at time } t$$
$$q_{B}(t) = \text{The probability that a prod.unit is in state } S_{B} \text{ at time } t$$
$$q_{\emptyset}(t) = \text{The probability that a prod.unit is in state } S_{\emptyset} \text{ at time } t$$

The distribution of $Y_3(t)$, $Y_4(t)$ and $Y_5(t)$ (cont.)

Note that we obviously have:

$$q_{AB}(t) + q_A(t) + q_B(t) + q_\emptyset(t) = 1$$

Thus, we can express $q_\emptyset(t)$ as $(1 - q_{AB}(t) - q_A(t) - q_B(t))$.

Since the subsystems have identical stochastic properties and are independent of each other, it follows that the vector $(Y_3(t), Y_4(t), Y_5(t))$ is multinomially distributed. That is, we have:

$$P(Y_3(t) = y_3, Y_4(t) = y_4, Y_5(t) = y_5) = \frac{5!}{y_3!y_4!y_4!(5 - y_3 - y_4 - y_5)!}$$

$$\cdot [q_{AB}(t)]^{y_3} \cdot [q_A(t)]^{y_4} \cdot [q_B(t)]^{y_5} \cdot [q_\emptyset(t)]^{5-y_3-y_4-y_5}$$

where $y_3 = 0, 1, \ldots 5$, $y_4 = 0, 1, \ldots (5 - y_3)$ and $y_5 = 0, 1, \ldots (5 - y_3 - y_4)$.

The probabilities $q_{AB}(t)$, $q_A(t)$ and $q_B(t)$

In order to find the the probabilities $q_{AB}(t)$, $q_A(t)$ and $q_B(t)$ we introduce the following events:

$$E_A = \text{The event that the production unit communicates with } A \text{ at time } t$$

$$E_B = \text{The event that the production unit communicates with } B \text{ at time } t$$

We may then express the desired probabilities as:

$$q_{AB}(t) = P(E_A \cap E_B)$$

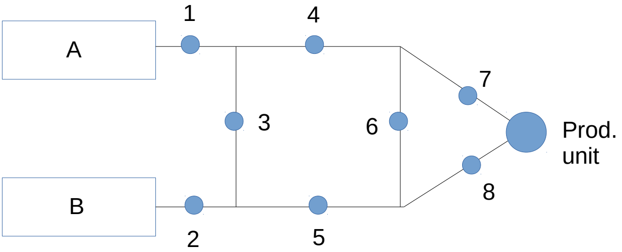
$$q_A(t) = P(E_A) - P(E_A \cap E_B)$$

$$q_B(t) = P(E_B) - P(E_A \cap E_B)$$

Thus, we have reduced the problem to computing the probabilities $P(E_A)$, $P(E_B)$ and $P(E_A \cap E_B)$.

The probabilities $q_{AB}(t)$, $q_A(t)$ and $q_B(t)$ (cont.)

In the following we simplify the notation by omitting the time t, and consider one of the subsystems:



We also introduce state variables for the 8 components in this system and denote these by $Z_1, \ldots, Z_8$. By previous assumptions we know that the state variables are independent and that:

$$P(Z_i = 1) = q_i, \quad i = 1, \ldots, 8.$$

Computing $P(E_A \cap E_B)$

In order to compute the probability $P(E_A \cap E_B)$ we condition on the states of the two bridges, i.e., components 3 and 6, and get:

$$\begin{aligned}
P(E_A \cap E_B) &= P(E_A \cap E_B \mid Z_3 = 1, Z_6 = 1) \cdot q_3 q_6 \\
&+ P(E_A \cap E_B \mid Z_3 = 1, Z_6 = 0) \cdot q_3(1 - q_6) \\
&+ P(E_A \cap E_B \mid Z_3 = 0, Z_6 = 1) \cdot (1 - q_3) q_6 \\
&+ P(E_A \cap E_B \mid Z_3 = 0, Z_6 = 0) \cdot (1 - q_3)(1 - q_6)
\end{aligned}$$

It is easy to show that the conditional probabilities are:

$$\begin{aligned}
P(E_A \cap E_B \mid Z_3 = 1, Z_6 = 1) &= q_1 q_2 [q_4 + q_5 - q_4 q_5][q_7 + q_8 - q_7 q_8] \\
P(E_A \cap E_B \mid Z_3 = 1, Z_6 = 0) &= q_1 q_2 [q_4 q_7 + q_5 q_8 - q_4 q_5 q_7 q_8] \\
P(E_A \cap E_B \mid Z_3 = 0, Z_6 = 1) &= q_1 q_2 q_4 q_5 [q_7 + q_8 - q_7 q_8] \\
P(E_A \cap E_B \mid Z_3 = 0, Z_6 = 0) &= q_1 q_2 q_4 q_5 q_7 q_8
\end{aligned}$$

Computing $P(E_A)$ and $P(E_B)$

We observe that for the event E_A Component 2 is irrelevant. When this component is removed, we note that Component 3 and Component 5 are in series. From this it is easy to show that:

$$P(E_A) = q_1 q_6 [q_4 + q_3 q_5 - q_3 q_4 q_5][q_7 + q_8 - q_7 q_8] \\ + q_1 (1 - q_6)[q_4 q_7 + q_3 q_5 q_8 - q_3 q_4 q_5 q_7 q_8]$$

We observe that for the event E_B Component 1 is irrelevant. When this component is removed, we note that Component 3 and Component 4 are in series. From this it is easy to show that:

$$P(E_B) = q_2 q_6 [q_5 + q_3 q_4 - q_3 q_4 q_5][q_7 + q_8 - q_7 q_8] \\ + q_2 (1 - q_6)[q_5 q_8 + q_3 q_4 q_7 - q_3 q_4 q_5 q_7 q_8]$$

The distribution of $Y_3(t)$, $Y_4(t)$ and $Y_4(t)$ (cont.)

Having computed the probabilities $P(E_A)$, $P(E_B)$ and $P(E_A \cap E_B)$, we can compute the probabilities $q_{AB}(t)$, $q_A(t)$ and $q_B(t)$.

Furthermore, given $q_{AB}(t)$, $q_A(t)$ and $q_B(t)$ we have what we need to compute the distribution of the vector $(Y_3(t), Y_4(t), Y_5(t))$.

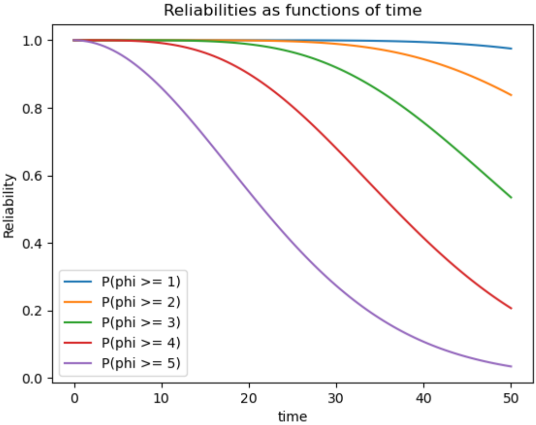
Since the system state $\phi(t)$ is expressed in terms of $Y_1(t), Y_2(t), Y_3(t), Y_4(t), Y_5(t)$, we can find the distribution of $\phi(t)$ by a total state space enumeration of these input variables.

- The number of states for $Y_1(t)$ is 7
- The number of states for $Y_2(t)$ is 7
- The number of states for the vector $(Y_3(t), Y_4(t), Y_5(t))$ is 56.

Thus, the total number of states needed in the enumeration is $7 \cdot 7 \cdot 56 = 2744$. While this is still a large number of terms, it is much smaller than $2^{52} \approx 4.504 \cdot 10^{15}$.

The computer we used

The results


Reliabilities as functions of time
Reliability
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
time
P(phi >= 1)
P(phi >= 2)
P(phi >= 3)
P(phi >= 4)
P(phi >= 5)