STK3405 – Lecture 7

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Network systems

Definition (Network systems)

A *network system* is a binary monotone system (C, ϕ) where the components are objects in a *network* (or *graph*), and where the system is functioning if the network satisfies some given *connectivity condition*, i.e., that some subset of the nodes, referred to as the *terminals* of the network, can communicate through the network.

In general a component can be a *node* or an *edge* in the network. In our setting, however, the nodes are typically assumed to be *perfect*. Thus, the component set C is typically equal to the set of edges in the graph.

Undirected network systems

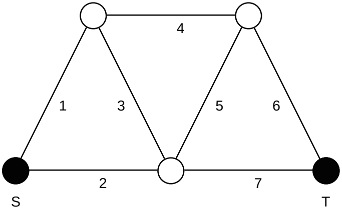


Figure: An undirected network with terminal nodes S and T.

The system is functioning if the terminals can communicate through the network.

NOTE: In undirected networks signals can be sent both ways through an edge.

Directed network systems

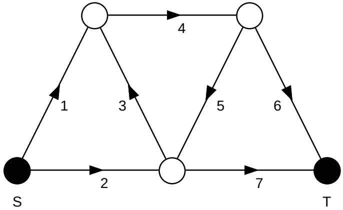
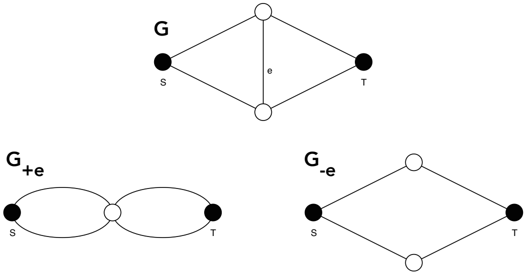


Figure: A directed network system with terminal nodes S and T.

The system is functioning if the terminals can communicate through the network.

NOTE: In directed networks signals can only be sent through an edge according to the direction of the edge.

Contraction and restriction



If G is an undirected or directed graph, and e is an edge in G, we define:

- The *contraction* of G with respect to e is the graph G_{+e} obtained from G by deleting e and merging its endnodes.
- The *restriction* of G with respect to e is the graph G_{-e} obtained from G by deleting e.

A graph H is said to be a *minor* of the graph G, if H can be obtained from G by performing a sequence of contractions and restrictions.

Contraction and restriction (cont.)

Undirected network systems

Let (C, ϕ) be an undirected network system with graph G, and let $e \in C$ be some edge in G. Then:

- G_{+e} represents the binary system obtained from (C, ϕ) by conditioning on that e is *functioning* (i.e. $x_e = 1$).
- G_{-e} represents the binary system obtained from (C, ϕ) by conditioning on that e is *failed* (i.e. $x_e = 0$).

The class of undirected network systems is *closed under pivotal decompositions*.

For an undirected network system pivotal decompositions corresponds to contractions and restrictions of the graph.

Contraction and restriction (cont.)

Directed network systems

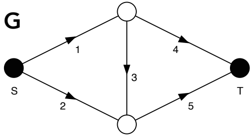
Let (C, ϕ) be an directed network system with graph G, and let $e \in C$ be some edge in G. Then:

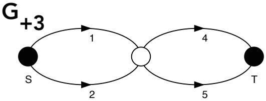
- G_{+e} does *not necessarily* represent the binary system obtained from (C, ϕ) by conditioning on that e is *functioning* (i.e. $x_e = 1$).
- G_{-e} represents the binary system obtained from (C, ϕ) by conditioning on that e is *failed* (i.e. $x_e = 0$).

The class of directed network systems is *not closed under pivotal decompositions*.

For a directed network system conditioning on that an edge e is functioning, may result in a system which is *not* representable as a network system.

Example – A directed bridge system





The minimal path sets of the directed network system G are:

$$P_1 = \{1,4\}, \quad P_2 = \{1,3,5\}, \quad P_3 = \{2,5\}.$$

Given that component 3 functions, we have the following minimal path sets:

$$P_1 = \{1,4\}, \quad P_2' = \{1,5\}, \quad P_3 = \{2,5\}.$$

This system can *not* be represented as a network system.

The minimal path sets of the network system G_{+3} are:

$$P_1 = \{1,4\}, \quad P_2' = \{1,5\}, \quad P_3 = \{2,5\}, \quad P_4 = \{2,4\}.$$

Computing the reliability of directed network systems

Directed network systems

Definition

A Source-to-K-terminal-system (SKT-system) is a system defined relative to a directed network where the system functions if and only if a node S (called the source) can send information to a given set of K nodes $T_1, \ldots, T_K$ (called the terminals).

The components of the system are the *directed edges* of the network, while the *nodes* are assumed to be *functioning perfectly* with probability one.

Directed network systems (cont.)

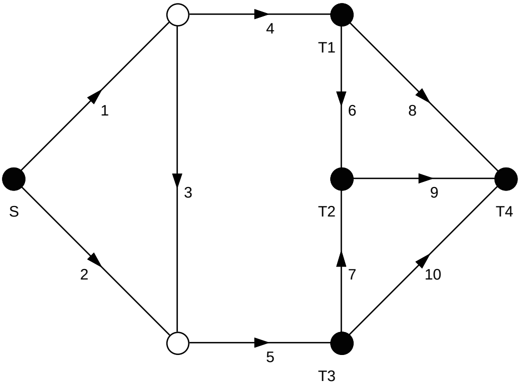


Figure: An S4T system with components $1, 2, \ldots, 10$. The node S is the source, while the nodes T_1, T_2, T_3, T_4 are the terminals.

Directed network systems (cont.)

Theorem

Let $\phi(\boldsymbol{X}) = \sum_{A \subseteq C} \delta(A) \prod_{i \in A} X_i$ *be the structure function of an SKT system.*

- *If* A *can be expressed as a union of minimal path sets, and the subgraph spanned by* A *does not contain any directed cycle, we have:*

$$\delta(A) = (-1)^{|A| - v(A) + 1}$$

  where $v(A)$ *denotes the number of nodes in the subgraph spanned by* A*.*

- *In the opposite case we have:*

$$\delta(A) = 0$$

Example of a directed network system

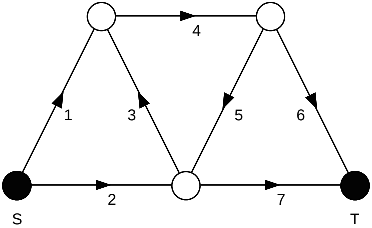


Figure: An S1T system (C, ϕ) where $C = \{1, \ldots, 7\}$.

The system is functioning if the source S can send signals to the terminal T through the network.

The component state variables are independent and $P(X_i = 1) = p_i$ for $i \in C$.

Example of a directed network system (cont.)

The minimal path sets of this system are $P_1 = \{1, 4, 6\}$, $P_2 = \{1, 4, 5, 7\}$, $P_3 = \{2, 3, 4, 6\}$ and $P_4 = \{2, 7\}$. We calculate the reliability of this system by using the inclusion-exclusion formula. Since there are 4 minimal path sets, this formula will consist of $2^4 - 1 = 15$ terms before we simplify:

$$
\begin{aligned}
h(\boldsymbol{p}) = p_1 p_4 p_6 &+ p_1 p_4 p_5 p_7 + p_2 p_3 p_4 p_6 + p_2 p_7 \\
&- p_1 p_4 p_5 p_6 p_7 - p_1 p_2 p_3 p_4 p_6 - p_1 p_2 p_4 p_6 p_7 \\
&- p_1 p_2 p_3 p_4 p_5 p_6 p_7 - p_1 p_2 p_4 p_5 p_7 - p_2 p_3 p_4 p_6 p_7 \\
&+ p_1 p_2 p_3 p_4 p_5 p_6 p_7 + p_1 p_2 p_4 p_5 p_6 p_7 \\
&+ p_1 p_2 p_3 p_4 p_6 p_7 + p_1 p_2 p_3 p_4 p_5 p_6 p_7 \\
&- p_1 p_2 p_3 p_4 p_5 p_6 p_7.
\end{aligned}
$$

Example of a directed network system (cont.)

By merging similar terms we obtain:

$$
\begin{aligned}
h(\boldsymbol{p}) = {} & p_1 p_4 p_6 + p_1 p_4 p_5 p_7 + p_2 p_3 p_4 p_6 + p_2 p_7 \\
& - p_1 p_4 p_5 p_6 p_7 - p_1 p_2 p_3 p_4 p_6 - p_1 p_2 p_4 p_6 p_7 \\
& - p_1 p_2 p_4 p_5 p_7 - p_1 p_2 p_4 p_5 p_7 \\
& + p_1 p_2 p_4 p_5 p_6 p_7 + p_1 p_2 p_3 p_4 p_6 p_7 .
\end{aligned}
$$

Here $\delta(A)$ is either $+1$, -1 or zero for all $A \subseteq C$.

Since the network contains a directed cycle $\{3, 4, 5\}$, $\delta(C) = 0$.

Example of a directed network system (cont.)

$$\begin{aligned}
\delta(\{1,4,6\}) &= (-1)^{3-4+1} = +1,\\
\delta(\{1,4,5,7\}) &= (-1)^{4-5+1} = +1,\\
&\cdots\cdots\\
\delta(\{1,4,5,6,7\}) &= (-1)^{5-5+1} = -1,\\
\delta(\{1,2,3,4,6\}) &= (-1)^{5-5+1} = -1,\\
\delta(\{1,2,4,6,7\}) &= (-1)^{5-5+1} = -1,\\
&\cdots\cdots\\
\delta(\{1,2,4,5,6,7\}) &= (-1)^{6-5+1} = +1,\\
\delta(\{1,2,3,4,6,7\}) &= (-1)^{6-5+1} = +1.
\end{aligned}$$

Linear consecutive k-out-of-n systems

Let (C, ϕ) be a linear consecutive 2-out-of-5 system where $C = \{1, \ldots, 5\}$.

Minimal path sets: $P_1 = \{1, 2\}$, $P_2 = \{2, 3\}$, $P_3 = \{3, 4\}$, $P_4 = \{4, 5\}$.

By using e.g., the inclusion-exclusion formula it is easy to see that the reliability of this system, assuming independent component state variables, is:

$$
\begin{aligned}
h(\boldsymbol{p}) = {} & p_1 p_2 + p_2 p_3 + p_3 p_4 + p_4 p_5 \\
& - p_1 p_2 p_3 - p_2 p_3 p_4 - p_3 p_4 p_5 - p_1 p_2 p_4 p_5 \\
& + p_1 p_2 p_3 p_4 p_5.
\end{aligned}
$$

It can be shown that this system is *not* an SKT-system.

Linear consecutive k-out-of-n systems share the property with SKT-systems that the signed domination function is either $+1$, -1 or zero for all subsets of the component set.

Section 4.5.

Computing the reliability of undirected network systems

Series and parallel reductions

We recall the following result:

Theorem (s-p-reductions)

Consider a binary monotone system (C, ϕ) and let $i, j \in C$, $i \neq j$.

- *If i and j are connected in series, then $h(\boldsymbol{p})$ will only depend on p_i and p_j through $p_i \cdot p_j$. Hence, i and j can be replaced by a single component with reliability $p_i \cdot p_j$ without altering the system reliability. Such a reduction is called a series reduction.*
- *If i and j are connected in parallel, then $h(\boldsymbol{p})$ will only depend on p_i and p_j through $p_i \amalg p_j$. Hence, i and j can be replaced by a single component with reliability $p_i \amalg p_j$ without altering the system reliability. Such a reduction is called a parallel reduction.*

s-p-reducible systems

Moreover, we have the following definition:

Definition

A system is *s-p-reducible* if there are components in either series or parallel in the system.

A system is *s-p-complex* if there are no components in either series or parallel in the system.

An *s-p system* is a system which can be s-p-reduced to a single component.

The factoring algorithm

Let (C, ϕ) be a binary monotone system. Assume that at least one of its components is relevant. To compute the reliability $h(\boldsymbol{p})$, proceed as follows:

Step 1: Perform all possible s-p-reductions. Let the reduced system be denoted by (C^r, ϕ^r). Then, (C^r, ϕ^r) must also have at least one relevant component.

Step 2: If (C^r, ϕ^r) contains precisely one relevant component with updated reliability p_e. Then, $h(\boldsymbol{p}) = p_e$.

If (C^r, ϕ^r) contains several relevant components, choose a component $e \in C^r$ and do a pivotal decomposition:

$$h(\boldsymbol{p}^{C^r}) = p_e h(1_e, \boldsymbol{p}^{C^r}) + (1 - p_e) h(0_e, \boldsymbol{p}^{C^r}).$$

Then, compute $h(1_e, \boldsymbol{p}^{C^r})$ and $h(0_e, \boldsymbol{p}^{C^r})$ by repeated use of the algorithm.

The factoring algorithm (cont.)

The factoring algorithm works best if the systems we need to handle along the way, can be represented efficiently, e.g., by a graph.

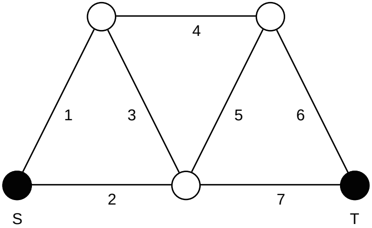
- Since the class of undirected network systems is closed under pivotal decompositions, the factoring algorithm works very well for such systems.
- Since the class of directed network systems is *not* closed under pivotal decompositions, the factoring algorithm does *not* work well for such systems.

NOTE: In general, the efficiency of the factoring algorithm depends on the choice of pivoting component.

In general it can be shown that when the factoring algorithm is applied to undirected network systems, one should always pivot such that both resulting substructures are *coherent*.

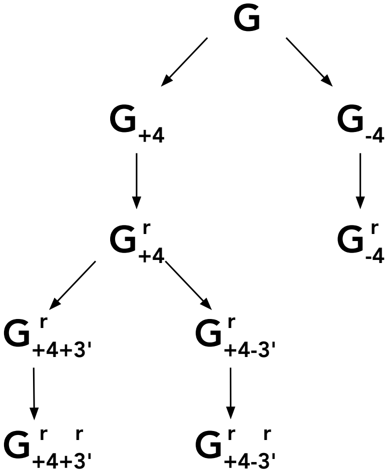
Example

Let (C, ϕ) be the undirected network system **G** shown below.

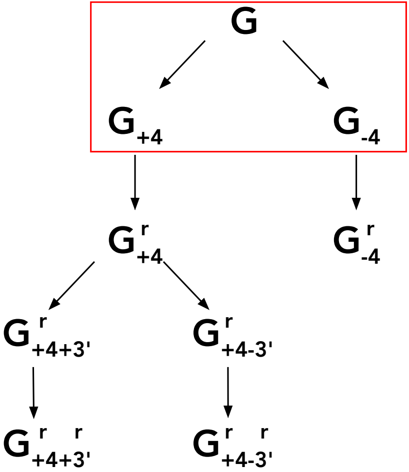


G is s-p-complex, so to proceed we must choose a component for pivotal decomposition.

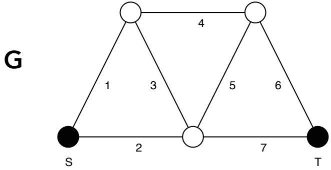
Example – Factoring tree



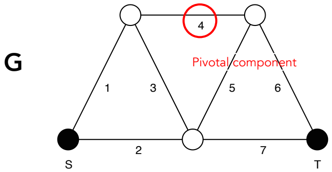
Example – Factoring tree (cont.)



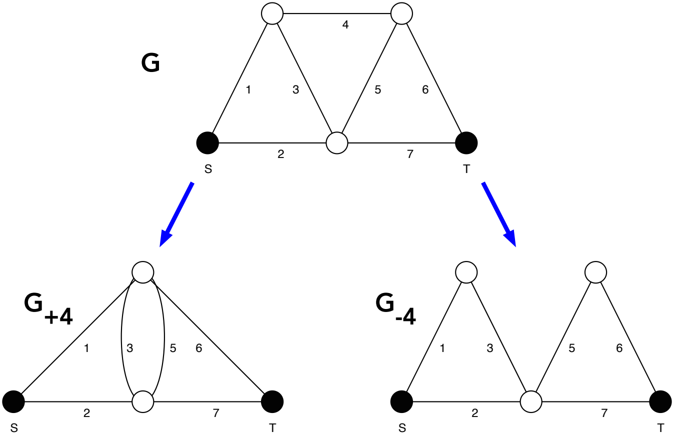
Example – Pivotal decomposition wrt. 4



Example – Pivotal decomposition wrt. 4 (cont.)

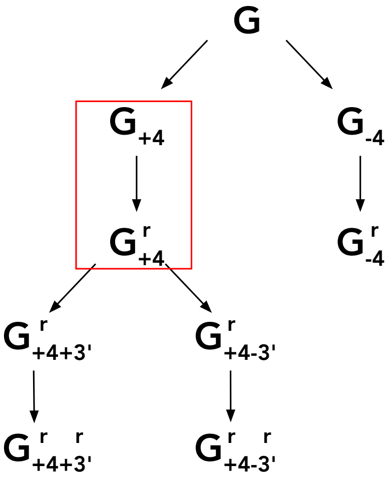


Example – Pivotal decomposition wrt. 4 (cont.)

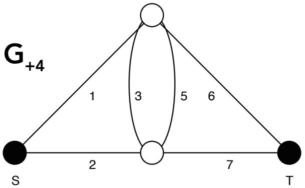


$$h(G) = p_4 \cdot h(G_{+4}) + (1 - p_4) \cdot h(G_{-4})$$

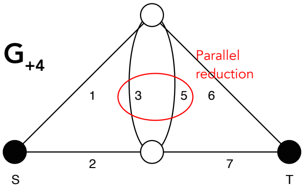
Example – Factoring tree (cont.)



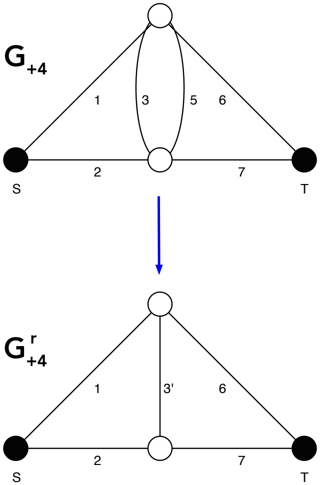
Example – Parallel reduction



Example – Parallel reduction (cont.)

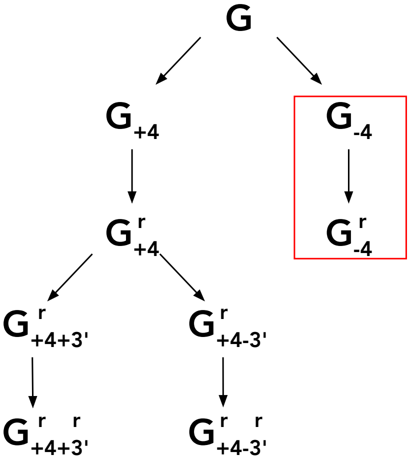


Example – Parallel reduction (cont.)

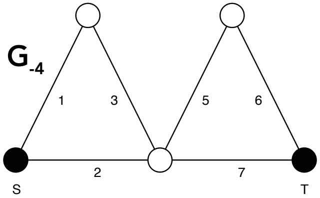


$$p_{3'} = p_3 \amalg p_5$$

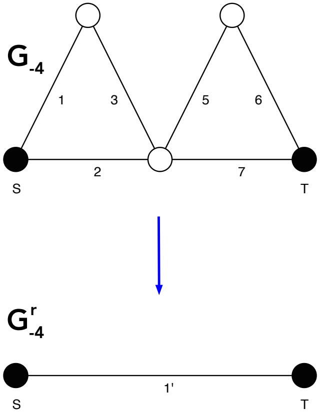
Example – Factoring tree (cont.)



Example – s-p-reducable system

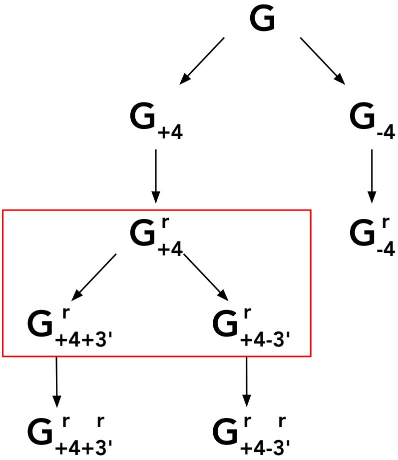


Example – s-p-reducable system (cont.)

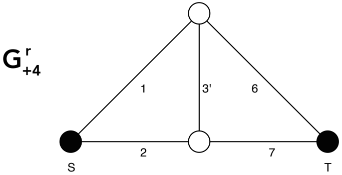


$$p_{1'} = h(G^r_{-4}) = [(p_1 p_3) \amalg p_2] \cdot [(p_5 p_6) \amalg p_7]$$

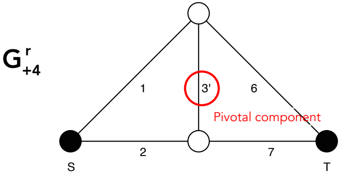
Example – Factoring tree (cont.)



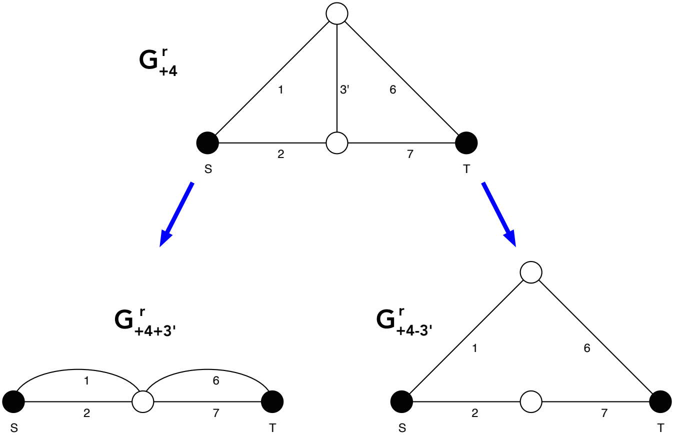
Example – Pivotal decomposition wrt. $3'$



Example – Pivotal decomposition wrt. $3'$ (cont.)

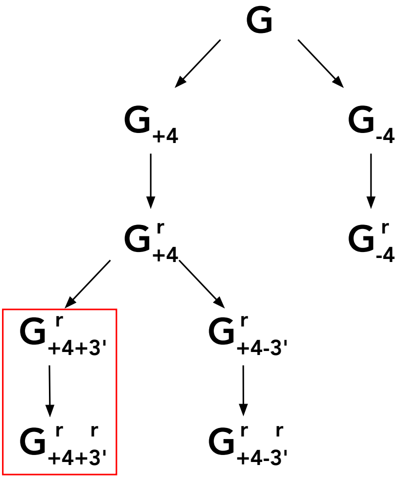


Example – Pivotal decomposition wrt. 3′ (cont.)

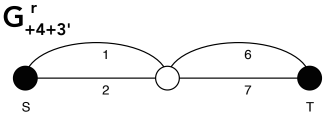


$$h(G^r_{+4}) = p_{3'} \cdot h(G^r_{+4+3'}) + (1 - p_{3'}) \cdot h(G^r_{+4-3'})$$

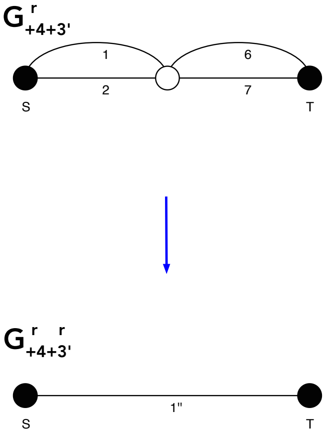
Example – Factoring tree (cont.)



Example – s-p-reducable system

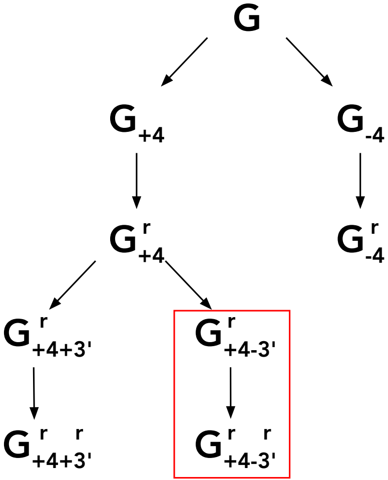


Example – s-p-reducable system (cont.)

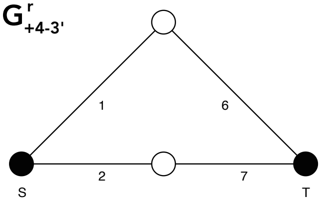


$$p_{1''} = h(G^{r}_{+4+^{r}3'}) = (p_1 \amalg p_2) \cdot (p_6 \amalg p_7)$$

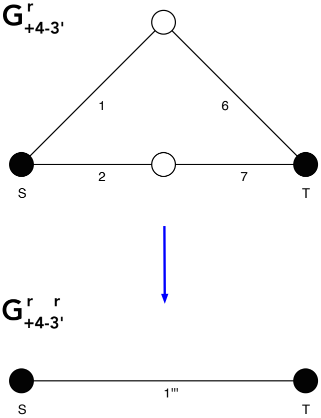
Example – Factoring tree



Example – s-p-reducable system



Example – s-p-reducable system (cont.)



$$p_{1'''} = h(G^r_{+4-3'}{}^r) = (p_1 \cdot p_6) \amalg (p_2 \cdot p_7)$$

Example – Summary

$$p_{1'} = h(G^{r}_{-4}) = [(p_1 p_3) \amalg p_2] \cdot [(p_5 p_6) \amalg p_7]$$

$$p_{1''} = h(G^{r}_{+4}{}^{r}_{+3'}) = (p_1 \amalg p_2) \cdot (p_6 \amalg p_7)$$

$$p_{1'''} = h(G^{r}_{+4}{}^{r}_{-3'}) = (p_1 \cdot p_6) \amalg (p_2 \cdot p_7)$$

$$p_{3'} = p_3 \amalg p_5$$

$$h = p_4 \cdot [p_{3'} p_{1''} + (1 - p_{3'}) p_{1'''}] + (1 - p_4) p_{1'}$$