STK3405 – Lecture 8

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Structural and reliability importance for components in binary monotone systems

Importance measures

- A measure of importance can be used to identify components that should be improved in order to increase the system reliability.
- A measure of importance can be used to identify components that most likely have failed, given that the system has failed.

Structural importance of a component

Criticality

Definition (Criticality)

Let (C, ϕ) *be a binary monotone system, and let* $i \in C$*. We say that component* i *is critical for the system if:*

$$\phi(1_i, \boldsymbol{x}) = 1 \text{ and } \phi(0_i, \boldsymbol{x}) = 0.$$

If this is the case, we also say that $(\cdot_i, \boldsymbol{x})$ *is a critical vector for component* i*.*

NOTE: Criticality is strongly related to the notion of relevance: A component i in a binary monotone system (C, ϕ) is relevant if and only if there exists at least one critical vector for i.

Criticality (cont.)

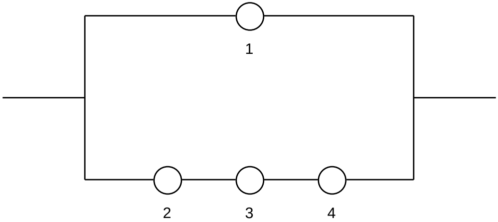

Figure: A binary monotone system (C, ϕ)

The structure function of the system (C, ϕ) is given by:

$$\phi(\boldsymbol{x}) = x_1 \amalg (x_2 \cdot x_3 \cdot x_4)$$

Criticality (cont.)

Component 1 is *critical* if $(\cdot_1, \boldsymbol{x})$ is:

$$(\cdot, 0, 0, 0), (\cdot, 1, 0, 0), (\cdot, 0, 1, 0), (\cdot, 0, 0, 1),$$

$$(\cdot, 1, 1, 0), (\cdot, 1, 0, 1), (\cdot, 0, 1, 1).$$

Component 2 is *critical* if $(\cdot_2, \boldsymbol{x}) = (0, \cdot, 1, 1)$,

Component 3 is *critical* if $(\cdot_3, \boldsymbol{x}) = (0, 1, \cdot, 1)$,

Component 4 is *critical* if $(\cdot_4, \boldsymbol{x}) = (0, 1, 1, \cdot)$.

Structural importance

Based on this Birnbaum suggested the following measure of structural importance of a component in a binary monotone system:

Definition (Structural importance)

Let (C, ϕ) be a binary monotone system of order n, and let $i \in C$. The Birnbaum measure for the structural importance of component i, denoted $J_B^{(i)}$, is defined as:

$$J_B^{(i)} = \frac{1}{2^{n-1}} \sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})].$$

Note that the denominator, 2^{n-1} is the total number of states for the $n - 1$ other components. Thus, $J_B^{(i)}$ can be interpreted as the fraction of all states for the $n - 1$ other components where component i is critical.

Structural importance

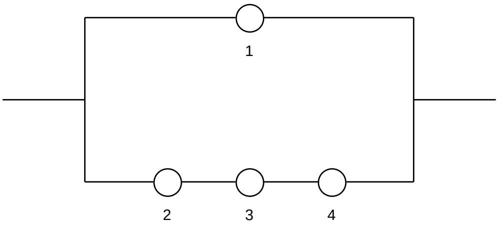

Figure: A binary monotone system (C, ϕ)

For this system we have the following structural importance measures:

$$J_B^{(1)} = \frac{7}{2^{4-1}} = \frac{7}{8}, \qquad J_B^{(2)} = J_B^{(3)} = J_B^{(4)} = \frac{1}{2^{4-1}} = \frac{1}{8}.$$

Structural importance

Let ϕ be a 2-out-of-3 system. To compute the structural importance of component 1, we note that the critical vectors for this component are $(\cdot, 1, 0)$ and $(\cdot, 0, 1)$. Hence, we have:

$$J_B^{(1)} = \frac{2}{2^{3-1}} = \frac{1}{2}.$$

By similar arguments, we find that:

$$J_B^{(2)} = J_B^{(3)} = \frac{1}{2}.$$

So in a 2-out-of-3 system, all of the components have the same structural importance. This is intuitively obvious since the structure function is symmetrical with respect to the components.

Reliability importance of a component

Reliability importance of a component

Definition (Reliability importance of a component)

Let (C, ϕ) be a binary monotone system, and let $i \in C$. Moreover, let $\boldsymbol{X}$ be the vector of component state variables.

The Birnbaum measure for the reliability importance of component i, denoted $I_B^{(i)}$ is defined as:

$$\begin{aligned} I_B^{(i)} &= P(\text{Component } i \text{ is critical for the system}) \\ &= P(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1). \end{aligned}$$

Reliability importance of a component (cont.)

Since the difference $\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})$ is a binary variable, it follows that:

$$I_B^{(i)} = E[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})] = E[\phi(1_i, \boldsymbol{X})] - E[\phi(0_i, \boldsymbol{X})].$$

In particular, if the component state variables of the system are independent, and $P(X_i = 1) = p_i$ for $i \in C$, we get that:

$$I_B^{(i)} = h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p}).$$

Reliability importance of a component (cont.)

Theorem (Partial derivative formula)

Let (C, ϕ) be a binary monotone system where the component state variables are independent, and $P(X_i = 1) = p_i$ for $i \in C$.

Then:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \qquad \text{for all } i \in C.$$

PROOF: By pivotal decomposition we have:

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p})$$

By differentiating this identity with respect to p_i we get:

$$\frac{\partial h(\boldsymbol{p})}{\partial p_i} = h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p}).$$

Hence, the result follows.

Reliability importance inequalities

Theorem (Reliability importance inequalities)

For a binary monotone system, (C, ϕ), we always have

$$0 \leq I_B^{(i)} \leq 1.$$

Assume that the component state variables are independent, and $P(X_j = 1) = p_j$, where $0 < p_j < 1$ for all $j \in C$.

If component i is relevant, we have:

$$0 < I_B^{(i)}.$$

Furthermore, if there exists at least one other relevant component, we also have:

$$I_B^{(i)} < 1.$$

Reliability importance inequalities (cont.)

PROOF: We note that the first inequality follows directly from the definition since the reliability importance is a *probability*.

We then assume that the component state variables are independent, and that $P(X_j = 1) = p_j$, where $0 < p_j < 1$ for all $j \in C$.

If component i is relevant, we know that h is strictly increasing in p_i.

That is, we must have:

$$\frac{\partial h(\boldsymbol{p})}{\partial p_i} > 0.$$

Combining this with the partial derivative formula, we get that $0 < I_B^{(i)}$.

Reliability importance inequalities (cont.)

Finally, we assume that there exists at least one other relevant component $k \in C$.

To show that this implies that $I_B^{(i)} < 1$, we assume instead that $I_B^{(i)} = 1$, and show that this leads to a contradiction.

By this assumption, it follows that :

$$P(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1) = 1$$

Since $0 < p_j < 1$, for all $j \in C$, it follows that $P((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})) > 0$ for all $(\cdot_i, \boldsymbol{x})$.

Hence, we must have that:

$$\phi(1_i, \boldsymbol{x}) = 1 \text{ and } \phi(0_i, \boldsymbol{x}) = 0 \text{ for all } (\cdot_i, \boldsymbol{x}).$$

Reliability importance inequalities (cont.)

At the same time, since component k is relevant, there exists a vector $(\cdot_k, \boldsymbol{y})$ such that:

$$\phi(1_k, \boldsymbol{y}) = 1 \text{ and } \phi(0_k, \boldsymbol{y}) = 0.$$

If $y_i = 1$, it follows that $\phi(1_i, 0_k, \boldsymbol{y}) = 0$, contradicting that $\phi(1_i, \boldsymbol{x}) = 1$ for all $(\cdot_i, \boldsymbol{x})$.

If $y_i = 0$, it follows that $\phi(0_i, 1_k, \boldsymbol{y}) = 1$, contradicting that $\phi(0_i, \boldsymbol{x}) = 0$ for all $(\cdot_i, \boldsymbol{x})$.

Hence, we conclude that for both possible values of y_i we end up with contradictions.

Thus, the only possibility is that $I_B^{(i)} < 1$.

Reliability importance and structural importance

Theorem (Reliability importance and structural importance)

Consider a binary monotone system (C, ϕ) where the component state variables are independent, and where $P(X_i = 1) = \frac{1}{2}$ for all $i \in C$. Then we have:

$$I_B^{(i)} = J_B^{(i)}$$

PROOF: If the component state variables are independent, and $P(X_i = 1) = \frac{1}{2}$ for all $i \in C$, we have:

$$P((\cdot_i, \boldsymbol{X}) = (\cdot_i, \boldsymbol{x})) = \prod_{j \neq i} P(X_j = x_j) = \prod_{j \neq i} (\frac{1}{2}) = \frac{1}{2^{n-1}}.$$

From this the result follows.

Reliability importance examples

In the following examples we consider binary monotone systems (C, ϕ) where $C = \{1, \ldots, n\}$.

We also assume that the component state variables are independent, and that:

$$P(X_i = 1) = p_i, \quad i \in C.$$

Without loss of generality we assume that the components are ordered so that:

$$p_1 \leq p_2 \leq \ldots \leq p_n. \tag{1}$$

Reliability importance examples (cont.)

Let (C, ϕ) be a series system. Then for all $i \in C$ we have:

$$I_B^{(i)} = \frac{\partial \prod_{j=1}^{n} p_j}{\partial p_i} = \prod_{j \neq i} p_j.$$

Hence, by the ordering (1), we get that:

$$I_B^{(1)} \geq I_B^{(2)} \geq \cdots \geq I_B^{(n)}.$$

Thus, in a series system the *worst* component, i.e., the one with the smallest reliability, has the greatest reliability importance.

Reliability importance examples (cont.)

Let (C, ϕ) be a parallel system. Then for all $i \in C$ we have:

$$I_B^{(i)} = \frac{\partial \coprod_{j=1}^n p_j}{\partial p_i} = \frac{\partial[1 - \prod_{j=1}^n (1 - p_j)]}{\partial p_i} = \prod_{j \neq i} (1 - p_j).$$

Hence, from the ordering (1)

$$I_B^{(1)} \leq I_B^{(2)} \leq \cdots \leq I_B^{(n)}.$$

Thus, in a parallel system the *best* component, i.e., the one with the greatest reliability, has the greatest reliability importance.

Reliability importance examples (cont.)

Let (C, ϕ) be a 2-out-of-3 system. It is then easy to show that:

$$\phi(\boldsymbol{X}) = X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3.$$

Hence, we have:

$$h(\boldsymbol{p}) = p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3.$$

This implies that:

$$I_B^{(1)} = \frac{\partial h(\boldsymbol{p})}{\partial p_1} = p_2 + p_3 - 2p_2p_3,$$

$$I_B^{(2)} = \frac{\partial h(\boldsymbol{p})}{\partial p_2} = p_1 + p_3 - 2p_1p_3,$$

$$I_B^{(3)} = \frac{\partial h(\boldsymbol{p})}{\partial p_3} = p_1 + p_2 - 2p_1p_2.$$

Reliability importance examples (cont.)

We then consider the function $f(p, q) = p + q - 2pq$ and note that:

$$I_B^{(1)} = f(p_2, p_3), \quad I_B^{(2)} = f(p_1, p_3), \quad I_B^{(3)} = f(p_1, p_2).$$

Moreover, the partial derivatives of f are respectively:

$$\frac{\partial f}{\partial p} = 1 - 2q, \qquad \frac{\partial f}{\partial q} = 1 - 2p.$$

If $p, q \leq \frac{1}{2}$, f is non-decreasing in p and q. Thus, if $p_1 \leq p_2 \leq p_3 \leq \frac{1}{2}$, we have:

$$f(p_1, p_2) \leq f(p_1, p_3) \leq f(p_2, p_3).$$

Hence, in this case we have:

$$I_B^{(3)} \leq I_B^{(2)} \leq I_B^{(1)}. \qquad (2)$$

Reliability importance examples (cont.)

If $p, q \geq \frac{1}{2}$, f is non-increasing in p and q. Thus, if $\frac{1}{2} \leq p_1 \leq p_2 \leq p_3$, we have:

$$f(p_2, p_3) \leq f(p_1, p_3) \leq f(p_1, p_2).$$

Hence, in this case we have:

$$I_B^{(1)} \leq I_B^{(2)} \leq I_B^{(3)}. \tag{3}$$

Reliability importance examples (cont.)

If $p_1 = \frac{1}{2} - z$, $p_2 = \frac{1}{2}$ and $p_3 = \frac{1}{2} + z$, where $z \in (0, \frac{1}{2})$, we get:

$$I_B^{(1)} = (\frac{1}{2}) + (\frac{1}{2} + z) - 2 \cdot (\frac{1}{2})(\frac{1}{2} + z) = \frac{1}{2},$$
$$I_B^{(2)} = (\frac{1}{2} - z) + (\frac{1}{2} + z) - 2 \cdot (\frac{1}{2} - z)(\frac{1}{2} + z) = \frac{1}{2} + 2z^2,$$
$$I_B^{(3)} = (\frac{1}{2} - z) + (\frac{1}{2}) - 2 \cdot (\frac{1}{2} - z)(\frac{1}{2}) = \frac{1}{2},$$

Hence in this case we have:

$$I_B^{(1)} = I_B^{(3)} \leq I_B^{(2)}. \tag{4}$$

Note that this result holds also if $z \in (-\frac{1}{2}, 0)$ in which case $p_1 > p_2 > p_3$.

Reliability importance examples (cont.)

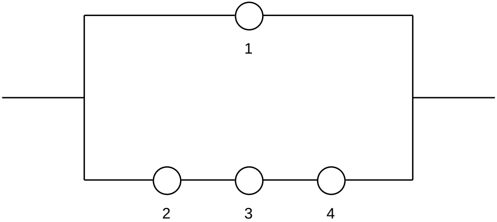

Figure: A binary monotone system (C, ϕ)

The structure function of this system is:

$$\phi(\boldsymbol{X}) = X_1 \amalg (X_2 \cdot X_3 \cdot X_4) = X_1 + X_2 \cdot X_3 \cdot X_4 - X_1 \cdot X_2 \cdot X_3 \cdot X_4$$

Thus, the reliability function is given by:

$$h(\boldsymbol{p}) = p_1 + p_2 \cdot p_3 \cdot p_4 - p_1 \cdot p_2 \cdot p_3 \cdot p_4$$

Reliability importance examples (cont.)

Hence we have:

$$I_B^{(1)} = 1 - p_2 \cdot p_3 \cdot p_4$$

$$I_B^{(2)} = p_3 \cdot p_4 - p_1 \cdot p_3 \cdot p_4 = (1 - p_1) \cdot p_3 \cdot p_4$$

$$I_B^{(3)} = p_2 \cdot p_4 - p_1 \cdot p_2 \cdot p_4 = (1 - p_1) \cdot p_2 \cdot p_4$$

$$I_B^{(4)} = p_2 \cdot p_3 - p_1 \cdot p_2 \cdot p_3 = (1 - p_1) \cdot p_2 \cdot p_3$$

If $p_1 = p_2 = p_3 = p_4 = p \in (0, 1)$, we have:

$$I_B^{(1)} = 1 - p^3$$

$$I_B^{(i)} = p^2 - p^3 < I_B^{(1)}, \quad i = 2, 3, 4.$$

Reliability importance examples (cont.)

Assume instead that $p_1 = 0.1$ and that $p_2 = p_3 = p_4 = 0.9$. Then we get:

$$I_B^{(1)} = 1 - p_2 \cdot p_3 \cdot p_4 = 1 - 0.9^3 = 0.271$$

$$I_B^{(2)} = p_3 \cdot p_4 - p_1 \cdot p_3 \cdot p_4 = (1 - p_1) \cdot p_3 \cdot p_4 = 0.9^3 = 0.729$$

$$I_B^{(3)} = p_2 \cdot p_4 - p_1 \cdot p_2 \cdot p_4 = (1 - p_1) \cdot p_2 \cdot p_4 = 0.9^3 = 0.729$$

$$I_B^{(4)} = p_2 \cdot p_3 - p_1 \cdot p_2 \cdot p_3 = (1 - p_1) \cdot p_2 \cdot p_3 = 0.9^3 = 0.729$$

Thus, in this case we have:

$$I_B^{(1)} < I_B^{(2)} = I_B^{(3)} = I_B^{(4)}.$$

The Barlow-Proschan and Natvig measures of reliability importance

Time-independent importance measures

Let (C, ϕ) be a binary monotone system where $C = \{1, \ldots, n\}$, and introduce:

$$X_i(t) = I(\text{Component } i \text{ is functioning at time } t), \quad i \in C.$$

The *Birnbaum measure* for reliability importance is based on the joint distribution of $X_1(t), \ldots, X_n(t)$:

$$P(X_1(t) = x_1, \ldots, X_n(t) = x_n)$$

What if we want to analyse the importance of the components not just for a given point of time t, but over the entire potential lifetime of the system?

NOTE: Throughout Chapter 5 we assume that the components are not repaired.

Section 5.3

The Barlow-Proschan measure of reliability importance

The Barlow-Proschan measure of reliability importance

Definition (Barlow-Proschan measure)

Let (C, ϕ) *be a non-trivial binary monotone system where* $C = \{1, \ldots, n\}$*. Moreover, let* T_i *denote the lifetime of component* i*,* $i \in C$*, and let* S *denote the lifetime of the system.*

The Barlow-Proschan measure of the reliability importance of component $i \in C$ *is defined as:*

$$\begin{aligned} I_{B-P}^{(i)} &= P(\text{Component } i \text{ fails at the same time as the system}) \\ &= P(T_i = S). \end{aligned}$$

Lebesgue measure

If $a < b$, the *length* of the set $[a, b]$ is $m_1([a, b]) = (b - a)$.

The definition of the function m_1 can be extended in a unique way to any (measurable) subset $A \subseteq \mathbb{R}$. The function m_1 is called *the Lebesgue measure* in $\mathbb{R}$.

If $A \subseteq \mathbb{R}$ is either a finite set or a countable set, it can be shown that $m_1(A) = 0$.

Lebesgue measure (cont.)

If $a_i < b_i$, $i = 1, \ldots, n$, the *volume* of the set $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is $m_n([a_1, b_1] \times \cdots \times [a_n, b_n]) = (b_1 - a_1) \cdots (b_n - a_n)$.

The definition of the function m_n can be extended in a unique way to any (measurable) subset $A \subseteq \mathbb{R}^n$. The function m_n is called *the Lebesgue measure* in $\mathbb{R}^n$.

If $A \subseteq \mathbb{R}^n$ has lower dimension than n (like e.g., a hyperplane), it can be shown that $m_n(A) = 0$.

Absolute continuity

- A real-valued stochastic variable, $T \in \mathbb{R}$ has an absolutely continuous distribution if $P(T \in A) = 0$ for all measurable sets $A \subseteq \mathbb{R}$ such that $m_1(A) = 0$.
- A vector-valued stochastic variable, $\boldsymbol{T} \in \mathbb{R}^n$ has an absolutely continuous distribution if $P(\boldsymbol{T} \in A) = 0$ for all measurable sets $A \subseteq \mathbb{R}^n$ such that $m_n(A) = 0$.
- If $T_1, \ldots, T_n$ are independent and absolutely continuously distributed, then $\boldsymbol{T} = (T_1, \ldots, T_n)$ is absolutely continuously distributed in $\mathbb{R}^n$.
- In particular, if $A = \{\boldsymbol{t} : t_i = t_j\}$, where $i \neq j$, then $m_n(A) = 0$. Hence, $P(T_i = T_j) = 0$ when $i \neq j$.

The Barlow-Proschan measure of reliability importance (cont.)

Theorem (Probability of system failure)

Let (C, ϕ) be a non-trivial binary monotone system where $C = \{1, \ldots, n\}$. Moreover, let T_i denote the lifetime of component i, $i \in C$, and let S denote the lifetime of the system.

Assume that $T_1, \ldots, T_n$ are independent and absolutely continuously distributed.

Then S is absolutely continuously distributed as well, and we have:

$$\sum_{i=1}^{n} I_{B-P}^{(i)} = 1.$$

The Barlow-Proschan measure of reliability importance (cont.)

PROOF: Since we have assumed that the system is non-trivial, the lifetime of the system, S can be expressed as:

$$S = \max_{1 \leq j \leq p} \min_{i \in P_j} T_i, \tag{5}$$

where $P_1, \ldots, P_p$ are the minimal path sets of the system. This implies that:

$$P(\bigcup_{i=1}^{n} \{T_i = S\}) = 1. \tag{6}$$

Let $A \subseteq \mathbb{R}$ be an arbitrary measurable set such that $m_1(A) = 0$. Since we have assumed that $T_1, \ldots, T_n$ are absolutely coninuously distributed, we get that:

$$0 \leq P(S \in A) \leq P(\bigcup_{i=1}^{n} \{T_i \in A\}) \leq \sum_{i=1}^{n} P(T_i \in A) = 0,$$

The Barlow-Proschan measure of reliability importance (cont.)

Since $T_1, \ldots, T_n$ are absolutely continuously distributed, the probability of having two or more components failing at the same time is zero.

This implies e.g., that $P(\{T_i = S\} \cap \{T_j = S\}) = 0$ for $i \neq j$. Thus, when calculating the probability of the union of the events $\{T_i = S\}$, $i = 1, \ldots, n$, all intersections can be ignored as they have zero probability of occurring.

Hence, by (6) we get:

$$1 = P(\bigcup_{i=1}^{n} \{T_i = S\}) = \sum_{i=1}^{n} P(T_i = S) = \sum_{i=1}^{n} I_{B-P}^{(i)},$$

where the second equality follows by ignoring all intersections of events $\{T_i = S\}$, $i = 1, \ldots, n$.

The last equality follows by the definition of $I_{B-P}^{(i)}$, and hence, the proof is complete.

The Barlow-Proschan measure of reliability importance (cont.)

Theorem (Integral formula for the Barlow-Proschan measure)

Let (C, ϕ) be a non-trivial binary monotone system where $C = \{1, \ldots, n\}$, and let T_i denote the lifetime of component i, $i \in C$.

Assume that $T_1, \ldots, T_n$ are independent, absolutely continuously distributed with densities $f_1, \ldots, f_n$ respectively. Then, we have:

$$I_{B-P}^{(i)} = \int_0^{\infty} I_B^{(i)}(t) f_i(t) dt,$$

where $I_B^{(i)}(t)$ denotes the Birnbaum measure of the reliability importance of component i at time t.

The Barlow-Proschan measure of reliability importance (cont.)

PROOF: From the definitions of the Barlow-Proschan measure and the Birnbaum measure, it follows that:

$$
\begin{aligned}
I_{B-P}^{(i)} &= P(\text{Component } i \text{ fails at the same time as the system}) \\
&= \int_0^\infty P(\text{Component } i \text{ is critical at time } t) \cdot f_i(t)dt \\
&= \int_0^\infty I_B^{(i)}(t) f_i(t)dt.
\end{aligned}
$$