UNIVERSITY OF OSLO

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Exam in:	STK3405/4405 — Introduction to risk and reliability analysis
Day of examination:	Monday December 12th 2022.

This problem set consists of 6 pages.

Appendices:	None.
Permitted aids:	Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

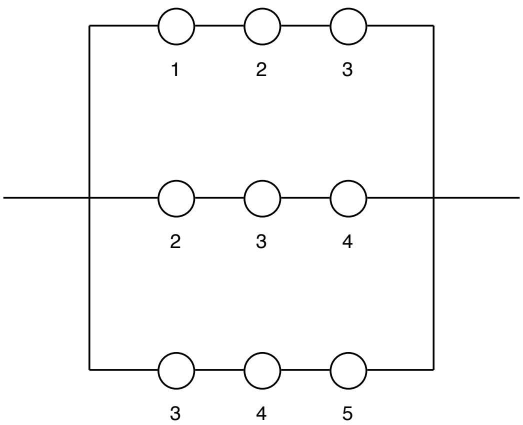

Figure 1: Block diagram a binary monotone system (C, ϕ)

Consider the binary monotone system (C, ϕ) shown in Figure 1. The component set of the system is $C = \{1, 2, 3, 4, 5\}$. Let $\boldsymbol{X} = (X_1, X_2, X_3, X_4, X_5)$ denote the vector of component state variables, and assume throughout this problem that X_1, X_2, X_3, X_4, X_5 are stochastically independent. Let $\boldsymbol{p} = (p_1, p_2, p_3, p_4, p_5)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, 3, 4, 5$. We also introduce the *reliability function*:

$$h(\boldsymbol{p}) = P(\phi(\boldsymbol{X}) = 1) = E[\phi(\boldsymbol{X})]$$

(Continued on page 2.)

a) Find the minimal path sets (3 sets) and the minimal cut sets (4 sets) of the system. Is the system *coherent*?

SOLUTION: Minimal path sets: $P_1 = \{1, 2, 3\}$, $P_2 = \{2, 3, 4\}$, $P_3 = \{3, 4, 5\}$. Minimal cut sets: $K_1 = \{1, 4\}$, $K_2 = \{2, 4\}$, $K_3 = \{2, 5\}$, $K_4 = \{3\}$.

Since $P_1 \cup P_2 \cup P_3 = C$, all components are contained in at least one minimal path set. Hence, it follows that the system is *coherent.*

Alternatively, since $K_1 \cup K_2 \cup K_3 \cup K_4 = C$, all components are contained in at least one minimal cut set. Hence, it follows that the system is *coherent.*

b) Show that the structure function of the system is given by:

$$\phi(\boldsymbol{X}) = X_1X_2X_3 + X_2X_3X_4 + X_3X_4X_5 - X_1X_2X_3X_4 - X_2X_3X_4X_5$$

Compute $h(\boldsymbol{p})$ given that $\boldsymbol{p} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

SOLUTION: Using the minimal path sets we get:

$$\phi(\boldsymbol{X}) = \coprod_{j=1}^{3} \prod_{i \in P_j} X_i = 1 - (1 - X_1X_2X_3)(1 - X_2X_3X_4)(1 - X_3X_4X_5)$$

$$= X_1X_2X_3 + X_2X_3X_4 + X_3X_4X_5$$
$$- X_1X_2X_3X_4 - X_2X_3X_4X_5 - X_1X_2X_3X_4X_5 + X_1X_2X_3X_4X_5$$

$$= X_1X_2X_3 + X_2X_3X_4 + X_3X_4X_5 - X_1X_2X_3X_4 - X_2X_3X_4X_5$$

The reliability of the system is then found by calculating $E[\phi(\boldsymbol{X})]$:

$$h(\boldsymbol{p}) = P(\phi(\boldsymbol{X}) = 1) = E[\phi(\boldsymbol{X})]$$

$$= E[X_1]E[X_2]E[X_3] + E[X_2]E[X_3]E[X_4] + E[X_3]E[X_4]E[X_5]$$
$$- E[X_1]E[X_2]E[X_3]E[X_4] - E[X_2]E[X_3]E[X_4]E[X_5]$$

$$= p_1p_2p_3 + p_2p_3p_4 + p_3p_4p_5 - p_1p_2p_3p_4 - p_2p_3p_4p_5$$

We then insert $p_i = \frac{1}{2}$ and get:

$$h(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = 3 \cdot \tfrac{1}{8} - 2 \cdot \tfrac{1}{16} = \tfrac{3-1}{8} = \tfrac{1}{4}.$$

The Birnbaum measure of the *reliability importance* of component i is defined as:

$$I_B^{(i)} = P(\text{Component } i \text{ is critical for the system}), \quad i \in C.$$

c) Show that:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \quad i \in C.$$

(Continued on page 3.)

SOLUTION: Using pivotal decomposition we have:

$$h(\boldsymbol{p}) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i) h(0_i, \boldsymbol{p})$$

We then take the partial derivative of $h(\boldsymbol{p})$ with respect to p_i and get:

$$\begin{aligned}
\frac{\partial h(\boldsymbol{p})}{\partial p_i} &= h(1_i, \boldsymbol{p}) - h(0_i, \boldsymbol{p}) \\
&= E[\phi(1_i, \boldsymbol{X})] - E[\phi(0_i, \boldsymbol{X})] = E[\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X})] \\
&= P(\phi(1_i, \boldsymbol{X}) - \phi(0_i, \boldsymbol{X}) = 1) \\
&= P(\text{Component } i \text{ is critical for the system}) = I_B^{(i)},
\end{aligned}$$

where the second equality follows since the component state variables are assumed to be stochastically independent.

d) Use the result from (c) to show that:

$$\begin{aligned}
I_B^{(1)} &= p_2 p_3 (1 - p_4) \\
I_B^{(2)} &= p_1 p_3 (1 - p_4) + p_3 p_4 (1 - p_5) \\
I_B^{(3)} &= p_1 p_2 (1 - p_4) + p_2 p_4 (1 - p_5) + p_4 p_5
\end{aligned}$$

SOLUTION: From (b) we know that:

$$h(\boldsymbol{p}) = p_1 p_2 p_3 + p_2 p_3 p_4 + p_3 p_4 p_5 - p_1 p_2 p_3 p_4 - p_2 p_3 p_4 p_5$$

By the result in (c) the Birnbaum measures for the components are given by the partial derivatives:

$$\begin{aligned}
I_B^{(1)} = \frac{\partial h(\boldsymbol{p})}{\partial p_1} &= p_2 p_3 - p_2 p_3 p_4 \\
&= p_2 p_3 (1 - p_4)
\end{aligned}$$

$$\begin{aligned}
I_B^{(2)} = \frac{\partial h(\boldsymbol{p})}{\partial p_2} &= p_1 p_3 + p_3 p_4 - p_1 p_3 p_4 - p_3 p_4 p_5 \\
&= p_1 p_3 (1 - p_4) + p_3 p_4 (1 - p_5)
\end{aligned}$$

$$\begin{aligned}
I_B^{(3)} = \frac{\partial h(\boldsymbol{p})}{\partial p_3} &= p_1 p_2 + p_2 p_4 + p_4 p_5 - p_1 p_2 p_4 - p_2 p_4 p_5 \\
&= p_1 p_2 (1 - p_4) + p_2 p_4 (1 - p_5) + p_4 p_5
\end{aligned}$$

e) Assume that $p_1 = p_2 = p_3 = p_4 = p_5 = p$, where $0 < p < 1$. Show that:

$$I_B^{(1)} = I_B^{(5)} < I_B^{(2)} = I_B^{(4)} < I_B^{(3)}$$

(Continued on page 4.)

SOLUTION: Using the results from (d) we get:

$$I_B^{(1)} = p^2(1-p)$$

$$I_B^{(2)} = 2p^2(1-p) = 2 \cdot I_B^{(1)}$$

$$I_B^{(3)} = 2p^2(1-p) + p^2 = I_B^{(2)} + p^2 = 2 \cdot I_B^{(1)} + p^2$$

By symmetry, we have:

$$I_B^{(5)} = I_B^{(1)} \quad \text{and} \quad I_B^{(4)} = I_B^{(2)}$$

Hence, we get the following ranking:

$$I_B^{(1)} = I_B^{(5)} < I_B^{(2)} = I_B^{(4)} < I_B^{(3)}$$

Problem 2

Let (C, ϕ) be a series system of non-repairable components where $C = \{1, \ldots, n\}$, and let:

$$X_i(t) = \begin{cases} 1 & \text{if component } i \text{ is functioning at time } t \\ 0 & \text{otherwise} \end{cases}$$

for $t \geq 0$ and $i \in C$. Moreover, let $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$. We assume that the component processes $\{X_1(t)\}_{t\geq 0}, \ldots, \{X_n(t)\}_{t\geq 0}$ are stochastically independent. The lifetime of component i is denoted by T_i, and we assume that the survival function of T_i is $\bar{F}_i$, for $i \in C$. Since the component processes are assumed to be stochastically independent, it follows that $T_1, \ldots, T_n$ are stochastically independent as well, and we have:

$$P(T_i > t) = \bar{F}_i(t), \quad i \in C, \quad t \geq 0,$$

a) Show that:

$$P(\phi(\boldsymbol{X}(t)) = 1) = \prod_{i=1}^{n} \bar{F}_i(t), \quad t \geq 0.$$

SOLUTION: We note that the event $\{X_i(t) = 1\}$ is equivalent to the event $\{T_i > t\}$, for $t \geq 0$ and $i \in C$. Hence, since the component processes are stochastically independent, we get for $t \geq 0$:

$$\begin{aligned} P(\phi(\boldsymbol{X}(t)) = 1) &= P(\prod_{i=1}^{n} X_i = 1) = \prod_{i=1}^{n} P(X_i(t) = 1) \\ &= \prod_{i=1}^{n} P(T_i > t) = \prod_{i=1}^{n} \bar{F}_i(t). \end{aligned}$$

We then introduce the *time dependent* Birnbaum measure of *reliability importance* of component $i \in C$ at time $t \geq 0$ defined as:

$$I_B^{(i)}(t) = P(\text{Component } i \text{ is critical for the system at time } t)$$

(Continued on page 5.)

b) Show that:

$$I_B^{(i)}(t) = \prod_{j \neq i} \bar{F}_j(t), \quad i \in C, \quad t \geq 0.$$

SOLUTION: By the definition of the Birnbaum measure we get:

$$\begin{aligned} I_B^{(i)}(t) &= P(\text{Component } i \text{ is critical for the system at time } t) \\ &= E[\phi(1_i, \boldsymbol{X}(t))] - E[\phi(0_i, \boldsymbol{X}(t))] \\ &= P(\prod_{j \neq i} X_j(t) = 1) - 0 = \prod_{j \neq i} P(X_j(t) = 1) \\ &= \prod_{j \neq i} P(T_j > t) = \prod_{j \neq i} \bar{F}_j(t) \end{aligned}$$

We then introduce the Barlow-Proschan measure of *reliability importance* of component $i \in C$ at time $t \geq 0$ defined as:

$$I_{B-P}^{(i)} = P(\text{Component } i \text{ fails at the same time as the system})$$

Assuming that $T_1, \ldots, T_n$ are absolutely continuously distributed with densities $f_1, \ldots, f_n$ respectively, it can be shown that we have:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) \cdot f_i(t) dt$$

Finally, we assume that:

$$\bar{F}_i(t) = e^{-\lambda_i t^\alpha}, \quad i \in C, \quad t \geq 0.$$

where $\alpha > 0$ and $\lambda_i > 0$, for all $i \in C$.

c) Show that:

$$I_{B-P}^{(i)} = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}, \quad i \in C.$$

SOLUTION: We let $R(t) = t^\alpha$ and note that since $\alpha > 0$, $R(t)$ is increasing, $R(0) = 0$ and $R(\infty) = \infty$. The survival functions of the lifetimes $T_1, \ldots, T_n$ can then be expressed as:

$$\bar{F}_i(t) = e^{-\lambda_i R(t)}, \quad i \in C, \quad t \geq 0.$$

Moreover, the densities of the lifetimes $T_1, \ldots, T_n$ can then be expressed as:

$$f_i(t) = -\frac{d\bar{F}_i(t)}{dt} = \lambda_i R'(t) \exp(-\lambda_i R(t)) = \lambda_i R'(t) \bar{F}_i(t)$$

(Continued on page 6.)

Furthermore, by the result in (b) we have:

$$I_B^{(i)}(t) = \prod_{j \neq i} \bar{F}_j(t)$$

Hence, the Barlow-Proschan measure becomes:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt = \int_0^\infty \prod_{j \neq i} \bar{F}_j(t) \lambda_i R'(t) \bar{F}_i(t) dt$$

$$= \int_0^\infty \prod_{j=1}^n \bar{F}_j(t) \lambda_i R'(t) dt = \lambda_i \int_0^\infty e^{-\sum_{j=1}^n \lambda_j R(t)} R'(t) dt$$

$$= \lambda_i \left[-\frac{1}{\sum_{j=1}^n \lambda_j} e^{-\sum_{j=1}^n \lambda_j R(t)} \right]_{t=0}^\infty = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}, \quad i \in C,$$

where the last equality follows since $R(0) = 0$ and $R(\infty) = \infty$ and $\lambda_i > 0$, for all $i \in C$.

ALTERNATIVE SOLUTION: The problem can also be solved without introducing the function $R(t)$ as follows:

We start out by finding the densities of the lifetimes $T_1, \ldots, T_n$:

$$f_i(t) = -\frac{d\bar{F}_i(t)}{dt} = \lambda_i \alpha t^{\alpha-1} e^{-\lambda_i t^\alpha} = \lambda_i \alpha t^{\alpha-1} \bar{F}_i(t)$$

Furthermore, by the result in (b) we have:

$$I_B^{(i)}(t) = \prod_{j \neq i} \bar{F}_j(t)$$

Hence, the Barlow-Proschan measure becomes:

$$I_{B-P}^{(i)} = \int_0^\infty I_B^{(i)}(t) f_i(t) dt = \int_0^\infty \prod_{j \neq i} \bar{F}_j(t) \lambda_i \alpha t^{\alpha-1} \bar{F}_i(t) dt$$

$$= \int_0^\infty \prod_{j=1}^n \bar{F}_j(t) \lambda_i \alpha t^{\alpha-1} dt = \lambda_i \int_0^\infty \alpha t^{\alpha-1} e^{-\sum_{j=1}^n \lambda_j t^\alpha} dt$$

We then observe that the last integrand is *almost* a density function of the same type as above. In order to get a proper density, we need to multiply it by the normalising factor $\sum_{i=1}^n \lambda_j$. At the same time we divide by the same factor outside the integral. The resulting integral then becomes an integral of a density over its outcome space $[0, \infty)$, which is equal to one. Hence:

$$I_{B-P}^{(i)} = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j} \int_0^\infty \sum_{j=1}^n \lambda_j \alpha t^{\alpha-1} e^{-\sum_{j=1}^n \lambda_j t^\alpha} dt = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}, \quad i \in C.$$

END