UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK3405/STK4405 — Elementary introduction to risk and reliability analysis.
Day of examination:	Wednesday 14. December 2016.
Examination hours:	15.00 – 19.00.

This problem set consists of 5 pages.

Appendices:	None.
Permitted aids:	Accepted calculator.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 13 subpoints will be equally weighted in the marking.

SOLUTION

Problem 1

a) Minimal path sets:

$$\{1,5\}, \{1,3,6\}, \{1,3,4,7\}, \{2,7\}, \{2,4,6\}, \{2,3,4,5\}$$

Minimal cut sets:

$$\{1,2\}, \{2,3,5\}, \{1,4,7\}, \{5,6,7\}, \{3,4,5,7\}, \{2,4,5,6\}, \{1,3,6,7\}$$

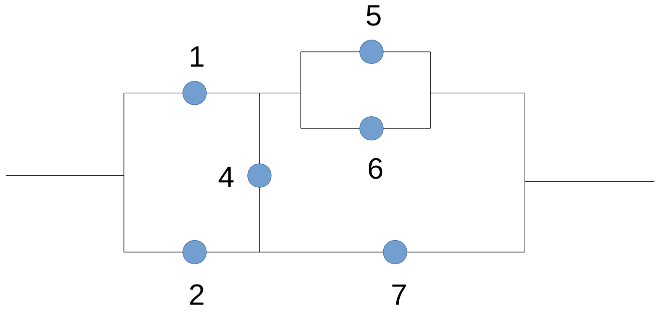
b) Since there are fewer minimal path sets than minimal cut sets, it is best to use the minimal path sets in the multiplication method. By doing this, we will in the best possible case get $2^6 - 1 = 63$ terms by using the multiplication method (before collecting terms). By total state enumeration, we get $2^7 - 1 = 127$ terms (since there are 7 components).

c) By pivoting with respect to the 3rd component, we get:

If component 3 is functioning:

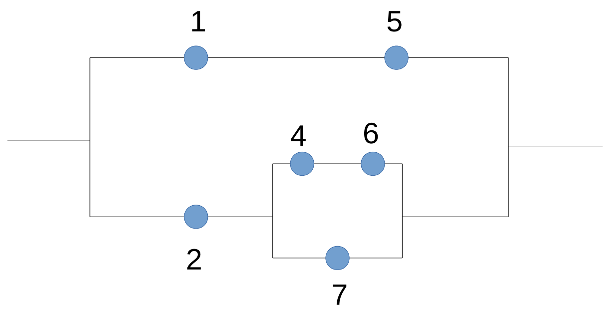
By parallel-reducing components 5 and 6, we see that this is a bridge structure. Hence, we can find the reliability by pivoting with respect to the bridge component, i.e. component 4.

(Continued on page 2.)



$$\begin{aligned} h(1_3, \boldsymbol{p}) &= p_4(p_1 \amalg p_2)((p_5 \amalg p_6) \amalg p_7) + (1-p_4)(p_1(p_5 \amalg p_6) \amalg p_2 p_7) \\ &= p_4(p_1 + p_2 - p_1 p_2)(p_5 + p_6 - p_5 p_6 + p_7 - p_7(p_5 + p_6 - p_5 p_6)) \\ &\quad + (1-p_4)(p_1(p_5 + p_6 - p_5 p_6) + p_2 p_7 - p_1 p_2 p_7 (p_5 + p_6 - p_5 p_6)) \end{aligned}$$

If component 3 *is not functioning:*



This is an s-p system where:

$$\begin{aligned} h(0_3, \boldsymbol{p}) &= (p_1 p_5) \amalg (p_2(p_4 p_6 \amalg p_7)) \\ &= p_1 p_5 \amalg p_2(p_4 p_6 + p_7 - p_4 p_6 p_7) \\ &= p_1 p_5 + p_2(p_4 p_6 + p_7 - p_4 p_6 p_7) - p_1 p_2 p_5 (p_4 p_6 + p_7 - p_4 p_6 p_7) \end{aligned}$$

Hence, the reliability function is:

$$h(\boldsymbol{p}) = p_3 h(1_3, \boldsymbol{p}) + (1-p_3) h(0_3, \boldsymbol{p})$$

d) We know that:

(Continued on page 3.)

$$I_B^{(3)} = \frac{\partial h(\boldsymbol{p})}{\partial p_3} = h(1_3, \boldsymbol{p}) - h(0_3, \boldsymbol{p}).$$

Hence, by using c), we see that

$$\begin{aligned} I_B^{(3)} &= p_4(p_1 + p_2 - p_1p_2)(p_5 + p_6 - p_5p_6 + p_7 - p_7(p_5 + p_6 - p_5p_6)) \\ &\quad +(1 - p_4)\big[p_1(p_5 + p_6 - p_5p_6) + p_2p_7 - p_1p_2p_7(p_5 + p_6 - p_5p_6)\big] \\ &\quad -\big[p_1p_5 + p_2(p_4p_6 + p_7 - p_4p_6p_7) - p_1p_2p_5(p_4p_6 + p_7 - p_4p_6p_7)\big] \end{aligned} \tag{1}$$

e) By using equation (1), we find that

$$\begin{aligned} J_B^{(3)} &= I_B^{(3)}|p_i = 1/2, i = 1, \dots, 7 \\ &= \big[h(1_3, \boldsymbol{p}) - h(0_3, \boldsymbol{p})\big]|p_i = 1/2, i = 1, \dots, 7 \end{aligned}$$

where

$$\begin{aligned} h(1_3, \boldsymbol{p})|p_i = 1/2, i = 1, \dots, 7 &= \tfrac{1}{2}\tfrac{3}{4}[\tfrac{3}{4} + \tfrac{1}{2} - \tfrac{3}{8}] + \tfrac{1}{2}[\tfrac{1}{2}\tfrac{3}{4} + \tfrac{1}{4} - \tfrac{1}{8}\tfrac{3}{4}] \\ &= \tfrac{3}{8}\tfrac{7}{8} + \tfrac{1}{2}(\tfrac{5}{8} - \tfrac{3}{32}) \\ &= \tfrac{21}{64} + \tfrac{17}{64} \\ &= \tfrac{38}{64} \end{aligned}$$

and

$$\begin{aligned} h(0_3, \boldsymbol{p})|p_i = 1/2, i = 1, \dots, 7 &= \tfrac{1}{4} + \tfrac{1}{2}(\tfrac{1}{4} + \tfrac{1}{2} - \tfrac{1}{8}) - \tfrac{1}{8}(\tfrac{1}{4} + \tfrac{1}{2} - \tfrac{1}{8}) \\ &= \tfrac{1}{4} + \tfrac{5}{16} - \tfrac{5}{64} \\ &= \tfrac{16}{64} + \tfrac{20}{64} - \tfrac{5}{64} = \tfrac{31}{64}. \end{aligned}$$

Hence,

$$J_B^{(3)} = \frac{38}{64} - \frac{31}{64} = \frac{7}{64} = \frac{7}{2^6}.$$

Alternatively:

$J_B^{(3)}$ = the number of critical path sets for the 3. component/(2^{7-1})

Since we have the following 7 critical path sets for component 3, the previous answer is verified:

$$\{1,3,6\}, \{1,3,4,7\}, \{2,3,4,5\}, \{1,3,6,7\}, \{1,3,4,6\}, \{1,3,4,6,7\}, \{1,2,3,6\}.$$

(Continued on page 4.)

Problem 2

a) See the proof of Theorem 3.4.2 in Natvig (1998).

b) See the first part of the proof of Theorem 3.4.7 in Natvig (1998).

c) See items *i*) − *iii*) and the comments thereafter on page 35 in Natvig (1998).

d) Note that

$$\begin{aligned} I_B^{(i)}(t) &= \tfrac{\partial h(\boldsymbol{p})}{\partial p_i} \\ &= h(1_i, \boldsymbol{p}(t)) - h(0_i, \boldsymbol{p}(t)) \\ &= E[\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t))] \\ &= \textstyle\sum_{(\cdot_i, \boldsymbol{x})} [\phi(1_i, \boldsymbol{x}) - \phi(0_i, \boldsymbol{x})] P(\boldsymbol{X}(t) = \boldsymbol{x}) \\ &= \textstyle\sum_{(\cdot_i, \boldsymbol{x})} 0 \cdot P(\boldsymbol{X}(t) = \boldsymbol{x}) = 0 \end{aligned}$$

where the second to last equality holds because component i is irrelevant.

By combining this with b), it follows that $I_{B-P}^{(i)} = 0$ as well.

So according to the Birnbaum and the Barlow-Prochan measures, the reliability importance of an irrelevant component is 0, which is intuitive.

However, for the Vesely-Fussel measure,

$$\begin{aligned} I_{V-F}^{(i)}(t) &= P[X_i(t) = 0 | \phi(\boldsymbol{X}(t)) = 0] \\ &= \tfrac{1}{P(\phi(\boldsymbol{X}(t))=0)} P[\phi(\boldsymbol{X}(t)) = 0 | X_i(t) = 0] P(X_i(t) = 0) \\ &= \tfrac{1}{P(\phi(\boldsymbol{X}(t))=0)} P[\phi(\boldsymbol{X}(t)) = 0] P(X_i(t) = 0) \\ &= P(X_i(t) = 0) \\ &= 1 - p_i(t) \end{aligned}$$

where the second equality follows from Bayes' formula, and the third follows from that component i is irrelevant. So if $p_i(t) \neq 1$, then $I_{V-F}^{(i)}(t) \neq 0$, which is unreasonable since component i is irrelevant.

Problem 3

a) See the proof of Theorem 3.6.1 in Natvig (1998).

b) See the proof of Theorem 3.6.5 in Natvig (1998).

c) Follows from a) and b): See the proof of Corollary 3.6.6 in Natvig (1998).

(Continued on page 5.)

d) We know from c) that

$$\max_{1 \le j \le p} \prod_{i \in P_j} p_i \le h.$$

By applying this to the dual structure function ϕ^D, we get

$$\max_{1 \le j \le p^D} \prod_{i \in P_j^D} p_i^D \le h^D.$$

Since the dual minimal path sets are the original minimal cut sets, this is equivalent to

$$\max_{1 \le j \le k} \prod_{i \in K_j} (1 - p_i) \le (1 - h).$$

So,

$$h \le 1 - \max_{1 \le j \le k} \prod_{i \in K_j} (1 - p_i) = \min_{1 \le j \le k} \big(1 - \prod_{i \in K_j} (1 - p_i)\big).$$

Hence,

$$h \le \min_{1 \le j \le k} \coprod_{i \in K_j} p_i$$

which is what we wanted to prove.

END