Exam problems

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Let (C, ϕ) be a binary monotone system with component set $C = \{1, \ldots, n\}$ and structure function ϕ. The Birnbaum measure of the reliability importance of component $i \in C$ at time $t \geq 0$ is defined as:

$$\begin{aligned} I_B^{(i)}(t) &= \text{P}(\text{Component } i \text{ is critical for the system at time } t) \\ &= \text{P}(\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t)) = 1) \\ &= \text{E}[\phi(1_i, \boldsymbol{X}(t)) - \phi(0_i, \boldsymbol{X}(t))], \end{aligned}$$

$\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t)) =$ the comp. state vector at time $t \geq 0$.

$\boldsymbol{p}(t) = (p_1(t), \ldots, p_n(t)) =$ the comp. reliability vector at time $t \geq 0$.

$h(t) = \text{P}(\phi(\boldsymbol{X}(t)) = 1) = \text{E}[\phi(\boldsymbol{X}(t)] =$ the sys. reliability at time $t \geq 0$.

If $X_1(t), \ldots, X_n(t)$ are independent, we write $h(t) = h(\boldsymbol{p}(t))$.

STK3405/4405 - 2017, problem 1 (a)

Let T_S denote the lifetime of the system, and let T_i denote the lifetime of component i, $i = 1, \ldots n$. Explain briefly that for $t \geq 0$ we have $T_S > t$ if and only if $\phi(\boldsymbol{X}(t)) = 1$.

SOLUTION: If $\phi(\boldsymbol{X}(t)) = 1$, this implies that the system is functioning at time t. Thus, the lifetime of the system, T_S must be greater than t.

If $\phi(\boldsymbol{X}(t)) = 0$, this implies that the system is failed at time t. Thus, the lifetime of the system, T_S must less than or equal to t.

Conclusion: $T_S > t$ if and only if $\phi(\boldsymbol{X}(t)) = 1$.

Use this to show that:

$$I_B^{(i)}(t) = \mathrm{P}(T_S > t | T_i > t) - \mathrm{P}(T_S > t | T_i \leq t),\ t \geq 0,\ i = 1, \ldots, n.$$

SOLUTION: By a similar argument we also have that $T_i > t$ if and only if $X_i(t) = 1$, $i = 1, \ldots, n$. Hence, we get:

$$\mathrm{E}[\phi(1_i, \boldsymbol{X}(t))] = \mathrm{P}(\phi(\boldsymbol{X}(t)) = 1 | X_i = 1) = \mathrm{P}(T_S > t | T_i > t)$$
$$\mathrm{E}[\phi(0_i, \boldsymbol{X}(t))] = \mathrm{P}(\phi(\boldsymbol{X}(t)) = 1 | X_i = 0) = \mathrm{P}(T_S > t | T_i \leq t)$$

By combining these equations we get:

$$\begin{aligned} I_B^{(i)}(t) &= \mathrm{E}[\phi(1_i, \boldsymbol{X}(t))] - \mathrm{E}[\phi(0_i, \boldsymbol{X}(t))] \\ &= \mathrm{P}(T_S > t | T_i > t) - \mathrm{P}(T_S > t | T_i \leq t) \end{aligned}$$

STK3405/4405 - 2017, problem 1 (b)

Assume that $X_1(t), \ldots, X_n(t)$ are stochastically independent. Show that we then have:

$$I_B^{(i)}(t) = \frac{\partial h(\boldsymbol{p}(t))}{\partial p_i(t)}, \quad t \geq 0, \quad i = 1, \ldots, n.$$

SOLUTION: If $X_1(t), \ldots, X_n(t)$ are stochastically independent, we have by using pivotal decomposition with respect to component i that:

$$\mathrm{P}(\phi(\boldsymbol{X}(t)) = 1) = h(\boldsymbol{p}(t)) = p_i(t)h(1_i, \boldsymbol{p}(t)) + (1 - p_i(t))h(0_i, \boldsymbol{p}(t))$$

Hence, by differentiating with respecting to $p_i(t)$ we get:

$$\begin{aligned}\frac{\partial h(\boldsymbol{p}(t))}{\partial p_i(t)} &= h(1_i, \boldsymbol{p}(t)) - h(0_i, \boldsymbol{p}(t)) \\ &= \mathrm{E}[\phi(1_i, \boldsymbol{X}(t))] - \mathrm{E}[\phi(0_i, \boldsymbol{X}(t))] = I_B^{(i)}(t).\end{aligned}$$

STK3405/4405 - 2017, problem 1 (c)

Explain briefly if $I_B^{(i,j)}(t) > 0$, this implies that:

$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 1_j, \boldsymbol{X}(t))] > \mathrm{E}[\phi(1_i, 0_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))],$$
$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(1_i, 0_j, \boldsymbol{X}(t))] > \mathrm{E}[\phi(0_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))],$$

while the opposite inequalities hold if $I_B^{(i,j)}(t) < 0$.

SOLUTION: Assume that $I_B^{(i,j)}(t) > 0$, that is:

$$\begin{aligned} I_B^{(i,j)}(t) = \mathrm{E}[&\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(1_i, 0_j, \boldsymbol{X}(t)) \\ &- \phi(0_i, 1_j, \boldsymbol{X}(t)) + \phi(0_i, 0_j, \boldsymbol{X}(t))] > 0. \end{aligned}$$

By adding $\mathrm{E}[\phi(1_i, 0_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))]$ on both sides of the inequality we get:

$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 1_j, \boldsymbol{X}(t))] > \mathrm{E}[\phi(1_i, 0_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))]$$

STK3405/4405 - 2017, problem 1 (c)

Similarly by adding $\mathrm{E}[\phi(0_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))]$ on both sides of the inequality we get:

$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(1_i, 0_j, \boldsymbol{X}(t))] > \mathrm{E}[\phi(0_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))]$$

If instead $I_B^{(i,j)}(t) < 0$, all inequalities are reversed.

STK3405/4405 - 2017, problem 1 (c)

Use this to give a practical interpretation of the sign of $I_B^{(i,j)}(t)$.

SOLUTION: If $I_B^{(i,j)}(t) > 0$, we have shown that:

$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 1_j, \boldsymbol{X}(t))] > \mathrm{E}[\phi(1_i, 0_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))]$$

Hence, component *i* is *more* important if component *j* is functioning than if component *j* is failed.

Similarly, if $I_B^{(i,j)}(t) > 0$, we have shown that:

$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(1_i, 0_j, \boldsymbol{X}(t))] > \mathrm{E}[\phi(0_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))]$$

Hence, component *j* is *more* important if component *i* is functioning than if component *i* is failed.

If $I_B^{(i,j)}(t) < 0$, we have shown that:

$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 1_j, \boldsymbol{X}(t))] < \mathrm{E}[\phi(1_i, 0_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))]$$

Hence, component *i* is *less* important if component *j* is functioning than if component *j* is failed.

Similarly, if $I_B^{(i,j)}(t) < 0$, we have shown that:

$$\mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t)) - \phi(1_i, 0_j, \boldsymbol{X}(t))] < \mathrm{E}[\phi(0_i, 1_j, \boldsymbol{X}(t)) - \phi(0_i, 0_j, \boldsymbol{X}(t))]$$

Hence, component *j* is *less* important if component *i* is functioning than if component *i* is failed.

Conclusion: If $I_B^{(i,j)}(t) > 0$, components *i* and *j* strengthen each others importance, while if $I_B^{(i,j)}(t) < 0$, components *i* and *j* weaken each others importance.

STK3405/4405 - 2017, problem 1 (d)

Show that for $i, j \in C$ and $t \geq 0$ we have:

$$I_B^{(i,j)}(t) = \mathrm{P}(T_S > t | T_i > t, T_j > t) - \mathrm{P}(T_S > t | T_i > t, T_j \leq t)$$
$$- \mathrm{P}(T_S > t | T_i \leq t, T_j > t) + \mathrm{P}(T_S > t | T_i \leq t, T_j \leq t).$$

SOLUTION: By the same argument as we used in (a) we have:

$$\mathrm{E}[\phi(1_i, 1_j \boldsymbol{X}(t))] = \mathrm{P}(T_S > t | T_i > t, T_j > t)$$
$$\mathrm{E}[\phi(1_i, 0_j \boldsymbol{X}(t))] = \mathrm{P}(T_S > t | T_i > t, T_j \leq t)$$
$$\mathrm{E}[\phi(0_i, 1_j \boldsymbol{X}(t))] = \mathrm{P}(T_S > t | T_i \leq t, T_j > t)$$
$$\mathrm{E}[\phi(0_i, 0_j \boldsymbol{X}(t))] = \mathrm{P}(T_S > t | T_i \leq t, T_j \leq t)$$

By combining all these equations we get the expression for $I_B^{(i,j)}(t)$.

STK3405/4405 - 2017, problem 1 (e)

Assume that $X_1(t), \ldots, X_n(t)$ are stochastically independent. Show that we then have:

$$I_B^{(i,j)}(t) = \frac{\partial^2 h(\boldsymbol{p}(t))}{\partial p_i(t) \partial p_j(t)}, \quad t \geq 0, \quad i, j = 1, \ldots, n.$$

SOLUTION: If $X_1(t), \ldots, X_n(t)$ are stochastically independent, we have by using pivotal decomposition with respect to components i and j that:

$$\begin{aligned} P(\phi(\boldsymbol{X}(t)) = 1) &= h(\boldsymbol{p}(t)) \\ &= p_i(t)p_j(t) \cdot h(1_i, 1_j, \boldsymbol{p}(t)) \\ &+ p_i(t)(1 - p_j(t)) \cdot h(1_i, 0_j, \boldsymbol{p}(t)) \\ &+ (1 - p_i(t))p_j(t) \cdot h(0_i, 1_j, \boldsymbol{p}(t)) \\ &+ (1 - p_i(t))(1 - p_j(t)) \cdot h(0_i, 0_j, \boldsymbol{p}(t)) \end{aligned}$$

STK3405/4405 - 2017, problem 1 (e)

Hence, by differentiating with respecting to $p_i(t)$ and $p_j(t)$ we get:

$$\frac{\partial^2 h(\boldsymbol{p}(t))}{\partial p_i(t)\partial p_j(t)} = h(1_i, 1_j, \boldsymbol{p}(t)) - h(1_i, 0_j, \boldsymbol{p}(t))$$

$$- h(0_i, 1_j, \boldsymbol{p}(t)) + h(0_i, 0_j, \boldsymbol{p}(t))$$

$$= \mathrm{E}[\phi(1_i, 1_j, \boldsymbol{X}(t))] - \mathrm{E}[\phi(1_i, 0_j, \boldsymbol{X}(t))]$$

$$- \mathrm{E}[\phi(0_i, 1_j, \boldsymbol{X}(t))] + \mathrm{E}[\phi(0_i, 0_j, \boldsymbol{X}(t))]$$

$$= I_B^{(i)}(t)$$

STK3405/4405 - 2017, problem 1 (f)

Show that $I_B^{(i,j)}(t) > 0$ if (C, ϕ) is a series system, while $I_B^{(i,j)}(t) < 0$ if (C, ϕ) is a parallel system. Give a brief comment to this result.

SOLUTION: If (C, ϕ) is a series system we have:

$$h(\boldsymbol{p}(t)) = \prod_{r=1}^{n} p_r(t)$$

By differentiating with respect to $p_i(t)$ and $p_j(t)$ we get:

$$I_B^{(i,j)}(t) = \frac{\partial^2 h(\boldsymbol{p}(t))}{\partial p_i(t) \partial p_j(t)} = \prod_{r \neq i,\, r \neq j} p_r(t) > 0$$

STK3405/4405 - 2017, problem 1 (f)

If (C, ϕ) is a series system we have:

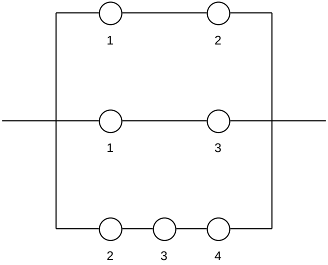
$$h(\boldsymbol{p}(t)) = \prod_{r=1}^{n} p_r(t) = 1 - \prod_{r=1}^{n}(1 - p_r(t))$$

By differentiating with respect to $p_i(t)$ and $p_j(t)$ we get:

$$I_B^{(i,j)}(t) = \frac{\partial^2 h(\boldsymbol{p}(t))}{\partial p_i(t) \partial p_j(t)} = -(-1)^2 \prod_{r \neq i,\, r \neq j} (1 - p_r(t)) < 0$$

COMMENT: In a series system a component is critical if and only if all the other components are functioning. Thus, in particular components i and j strengthen each others importance.
In a parallel system a component is critical if and only if all the other components are failed. Thus, in particular components i and j weaken each others importance.

STK3405/4405 - 2017, problem 2



The binary monotone system (C, ϕ) is shown in the block diagram in the figure above, where $C = \{1, 2, 3, 4\}$.

$\boldsymbol{X} = (X_1, X_2, X_3, X_4)$ denotes the vector of component state variables. X_1, X_2, X_3, X_4 are assumed independent.

$\boldsymbol{p} = (p_1, p_2, p_3, p_4)$ denotes the vector of component reliabilities. $p_i = P(X_i = 1)$ and $0 < p_i < 1$, $i = 1, 2, 3, 4$.

Find the minimal path and cut sets of (C, ϕ).

SOLUTION: By examining the reliability block diagram we find the following minmal path and cut sets:

MINIMAL PATH SETS: $\{1, 2\}$, $\{1, 3\}$, $\{2, 3, 4\}$.

MINIMAL CUT SETS: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$.

STK3405/4405 - 2017, problem 2 (b)

Show that the structure function of the system can be expressed as:

$$\phi(\boldsymbol{X}) = X_4[X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3] \\ + (1 - X_4)[X_1X_2 + X_1X_3 - X_1X_2X_3],$$

SOLUTION: We perform a pivotal decomposition with respect to component 4:

- Given that component 4 is *functioning*, the resulting system becomes a 2-out-of-3 system.
- Given that component 4 is *failed*, the resulting system becomes a system where component 1 is in series with the parallel connection of components 2 and 3.

STK3405/4405 - 2017, problem 2 (b)

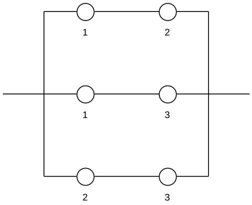


Figure: $\phi(1_4, \boldsymbol{X}) = X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3$

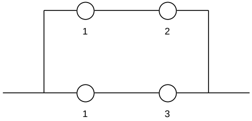


Figure: $\phi(0_4, \boldsymbol{X}) = X_1(X_2 + X_3 - X_2X_3)$

By combining these results we get:

$$\phi(\boldsymbol{X}) = X_4 \cdot \phi(1_4, \boldsymbol{X}) + (1 - X_4) \cdot \phi(0_4, \boldsymbol{X})$$
$$= X_4[X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3]$$
$$+ (1 - X_4)[X_1X_2 + X_1X_3 - X_1X_2X_3]$$

NOTE: In this case it is in fact easier to do a pivotal decomposition with respect to component 1 instead and get the simpler expression:

$$\phi(\boldsymbol{X}) = X_1 \cdot \phi(1_1, \boldsymbol{X}) + (1 - X_1) \cdot \phi(0_1, \boldsymbol{X})$$
$$= X_1[X_2 + X_3 - X_2X_3] + (1 - X_1)[X_2X_3X_4]$$

Use this to find the reliability of the system, $h(\boldsymbol{p}) = \mathrm{E}[\phi(\boldsymbol{X})]$.

SOLUTION: The reliability function is obtained from the structure function simply by replacing component state variables by their respective component reliabilities. Thus, we get:

$$
\begin{aligned}
h(\boldsymbol{p}) = p_4[p_1 p_2 + p_1 p_3 + p_2 p_3 - 2p_1 p_2 p_3] \\
+ (1 - p_4)[p_1 p_2 + p_1 p_3 - p_1 p_2 p_3]
\end{aligned}
$$

Or alternatively and simpler:

$$
h(\boldsymbol{p}) = p_1[p_2 + p_3 - p_2 p_3] + (1 - p_1)[p_2 p_3 p_4]
$$

STK3405/4405 - 2017, problem 2 (c)

You may use that the Birnbaum measure of the reliability importance of component $i \in C$ is given by:

$$I_B^{(i)} = \frac{\partial h(\boldsymbol{p})}{\partial p_i}, \quad i = 1, 2, 3, 4,$$

and that the Birnbaum measure of the joint reliability importance of the components $i, j \in C$ is given by:

$$I_B^{(i,j)} = \frac{\partial^2 h(\boldsymbol{p})}{\partial p_i \partial p_j}, \quad i, j = 1, 2, 3, 4.$$

STK3405/4405 - 2017, problem 2 (c)

Show that:

$$I_B^{(4)} = p_2 p_3 - p_1 p_2 p_3.$$

SOLUTION: By differentiating $h(\boldsymbol{p})$ with respect to p_4 we get:

$$I_B^{(4)} = \frac{\partial h(\boldsymbol{p})}{\partial p_3}$$

$$= [p_1 p_2 + p_1 p_3 + p_2 p_3 - 2 p_1 p_2 p_3]$$
$$- [p_1 p_2 + p_1 p_3 - p_1 p_2 p_3]$$

$$= p_2 p_3 - p_1 p_2 p_3.$$

STK3405/4405 - 2017, problem 2 (d)

Show that $I_B^{(1,4)} < 0$ and that $I_B^{(i,4)} > 0$, $i = 2, 3$.

SOLUTION: By differentiating $I_B^{(4)}$ with respect to p_1, p_2 and p_3 we get:

$$I_B^{(1,4)} = \frac{\partial}{\partial p_1}[p_2 p_3 - p_1 p_2 p_3] = -p_2 p_3 < 0$$

$$I_B^{(2,4)} = \frac{\partial}{\partial p_2}[p_2 p_3 - p_1 p_2 p_3] = p_3(1 - p_1) > 0$$

$$I_B^{(3,4)} = \frac{\partial}{\partial p_3}[p_2 p_3 - p_1 p_2 p_3] = p_2(1 - p_1) > 0$$

STK3405/4405 - 2017, problem 2 (d)

Give a brief comment to these results.

SOLUTION: We observe that $I_B^{(1,4)} < 0$. This indicates that components 1 and 4 weaken each others reliability importance.

In fact if component 1 is functioning, component 4 is irrelevant, while if component 1 is failed, component 4 has reliability importance $p_2 p_3$.

Moreover, we observe that $I_B^{(2,4)} > 0$, This indicates that components 2 and 4 strengthen each others reliability importance.

The same is true for components 3 and 4.

STK3405/4405 - 2017, problem 3

$\{S(t)\}$ is a stochastic process where $S(t)$ is the state of the process at time $t \geq 0$.

$\{S(t)\}$ is a *pure jump process* if $S(t)$ can be expressed as:

$$S(t) = S(0) + \sum_{j=1}^{\infty} \mathrm{I}(T_j \leq t) J_j, \qquad t \geq 0$$

We introduce:

$$N(t) = \sum_{j=1}^{\infty} I(T_j \leq t) = \text{The number of jumps in [0, t].}$$

$\{S(t)\}$ is *regular* if $P(N(t) < \infty) = 1$ for all $t > 0$.

STK3405/4405 - 2017, problem 3 (a)

Show that if the sequence $\{\Delta_j\}$ contains an infinite subsequence, $\{\Delta_{k_j}\}$, of independent, identically distributed stochastic variables such that $E[\Delta_{k_j}] = d > 0$, then $\{S(t)\}$ us regular.

SOLUTION: $\{S(t)\}$ is regular if and only if $T_\infty = \infty$ almost surely. Thus, $\{S(t)\}$ is regular if and only if $\sum_{j=1}^{\infty} \Delta_j$ is *divergent* with probability one.

By the strong law of large numbers it follows that:

$$P(\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \Delta_{k_j} = d) = 1.$$

This implies that the series $\sum_{j=1}^{\infty} \Delta_{k_j}$ is divergent with probability one.

Hence, since obviously $\sum_{j=1}^{\infty} \Delta_{k_j} \leq \sum_{j=1}^{\infty} \Delta_j = T_\infty$, the result follows.

STK3405/4405 - 2017, problem 3 (b)

Explain why regularity is important for simulations of pure jump processes.

SOLUTION: When we simulate a pure jump process over a given finite time interval $[0, T]$, typically using discrete event simulation, we need to be sure that the number of events we need to process is finite. If the pure jump process is regular, the number of such events is finite with probability one.