Solution exam 2018

Arne Bang Huseby and Kristina Rognlien Dahl

University of Oslo, Norway

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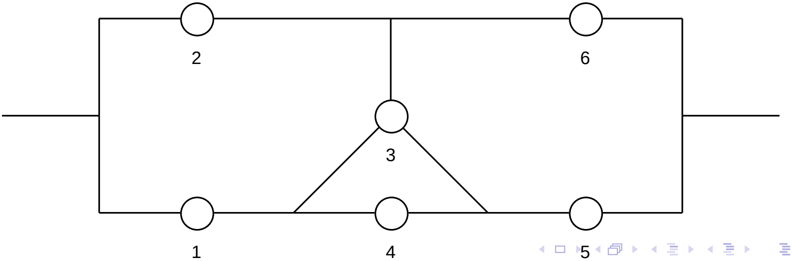
Problem 1

Consider the binary monotone system (C, ϕ) shown in the figure below. The component set of the system is $C = \{1, 2, \ldots, 6\}$. Let $\mathbf{X} = (X_1, X_2, \ldots, X_6)$ denote the vector of component state variables, and assume throughout this exercise that $X_1, X_2, \ldots, X_6$ are stochastically independent. Let $\mathbf{p} = (p_1, p_2, \ldots, p_6)$ denote the vector of component reliabilities, where $p_i = P(X_i = 1)$, $i = 1, 2, \ldots, 6$. We assume that $0 < p_i < 1$ for $i = 1, 2, \ldots, 6$.



Problem 1

a) Find the minimal path and cut sets of the system.

SOLUTION: Minimal path sets:

$$P_1 = \{1, 4, 5\}, P_2 = \{2, 6\}, P_3 = \{1, 3, 6\}, P_4 = \{2, 3, 5\}, P_5 = \{1, 3, 5\}$$

Minimal cut sets:

$$K_1 = \{1, 2\}, K_2 = \{5, 6\}, K_3 = \{2, 3, 4\}, K_4 = \{1, 3, 6\}, K_5 = \{2, 3, 5\}, K_6 = \{3, 4, 6\}$$

Problem 1

b) Use the result in a) to find an expression for the structure function of the system, and explain briefly how this can be used to find the system reliability. A detailed calculation is not required.

SOLUTION: The structure function can be expressed either as a series of the minimal cut parallel structures:

$$\phi(\mathbf{X}) \quad = \quad \prod_{i=1}^{6} \coprod_{j \in K_i} X_i \tag{1}$$

or as a parallel of the minimal path series structures:

$$\phi(\mathbf{X}) \quad = \quad \coprod_{i=1}^{5} \prod_{j \in P_i} X_i \tag{2}$$

Problem 1

The system reliability, $h(\mathbf{p})$, is defined as:

$$h(\mathbf{p}) = \mathrm{E}[\phi(\mathbf{X})].$$

Hence, one can find the system reliability by expanding either (1) or (2) into a sum of products of the component state variables. Using the independence of the components, the reliability of the system can then be found by replacing the component state variables X_i, $i = 1, 2, \ldots, 6$, with the corresponding component reliabilities $p_1, p_2, \ldots, p_6$.

Problem 1

c) Use the factoring algorithm to derive the reliability of the system in a different way from the one in b).

SOLUTION: By pivoting with respect to component 3, we see that the reliability $h(\boldsymbol{p})$ is:

$$h(\boldsymbol{p}) = p_3 h(1_3, \boldsymbol{p}) + (1 - p_3) h(1_3, \boldsymbol{p}).$$

We consider the two cases separately:

Component 3 *is functioning:*
In this case, the system becomes a series connection of two parallel systems (1 and 2 in parallel), (5 and 6 in parallel). Component 4 becomes irrelevant, see Figure 7.

Problem 1

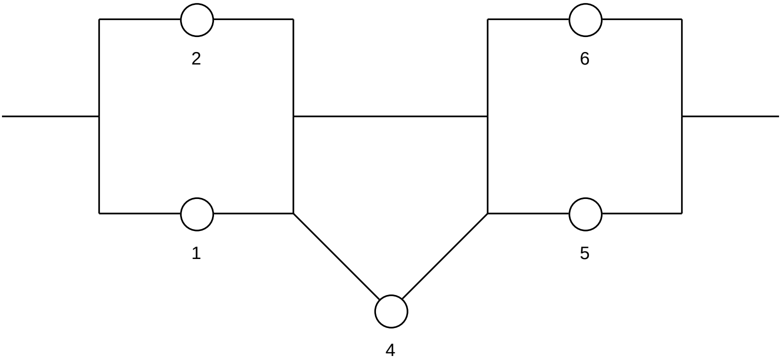


Figure: Resulting system if component 3 works.

Problem 1

Hence, the reliability function in this case is:

$$
\begin{aligned}
h(1_3, \boldsymbol{p}) &= (p_1 \coprod p_2)(p_5 \coprod p_6) \\
&= (p_1 + p_2 - p_1 p_2)(p_5 + p_6 - p_5 p_6).
\end{aligned}
$$

Problem 1

Component 3 *is not functioning:*
In this case, the system becomes a parallel connection of two series systems, (2 and 6 in series), (1, 4 and 5 in series), so the reliability function in this case is:

$$h(0_3, \boldsymbol{p}) = (p_2 p_6) \coprod (p_1 p_4 p_5)$$

$$= p_2 p_6 + p_1 p_4 p_5 - p_1 p_2 p_4 p_5 p_6.$$

Hence, the reliability function is:

$$h(\boldsymbol{p}) = p_3 h(1_3, \boldsymbol{p}) + (1 - p_3) h(0_3, \boldsymbol{p})$$

$$= p_3(p_1 + p_2 - p_1 p_2)(p_5 + p_6 - p_5 p_6) + (1 - p_3)(p_2 p_6 + p_1 p_4 p_5 - p_1 p_2 p_4 p_5 p_6).$$

Note that if you factor with respect to any of the other components, you will have to perform at least one additional pivot.

d) What is the definition of the Birnbaum measure for the reliability importance of a component?

SOLUTION: The Birnbaum measure for the reliability importance of a component is defined as follows. Let (C, ϕ) be a binary monotone system, and let $i \in C$. Moreover, let $\mathbf{X}$ be the vector of component state variables. The Birnbaum measure for the reliability importance of component i, denoted $I_B^{(i)}$ is defined as:

$$I_B^{(i)} := P(\text{Component } i \text{ is critical for the system})$$
$$= P(\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X}) = 1).$$

e) What is the reliability importance of the component 3 according to the Birnbaum measure? How can you use this result to find the structural importance of component 3?

SOLUTION: We know that:

$$I_B^{(3)} = \frac{\partial h(\boldsymbol{p})}{\partial p_3} = h(1_3, \boldsymbol{p}) - h(0_3, \boldsymbol{p}).$$

Hence, by using c), we see that

$$\begin{aligned} I_B^{(3)} &= (p_1 + p_2 - p_1 p_2)(p_5 + p_6 - p_5 p_6) - p_2 p_6 - p_1 p_4 p_5 \\ &\quad + p_1 p_2 p_4 p_5 p_6. \end{aligned} \tag{3}$$

Problem 1

By using equation (3), we find that:

$$\begin{aligned} J_B^{(3)} &= I_B^{(3)}\big|_{p_i=1/2} \quad i = 1, \ldots, 6 \\ &= \big[h(1_3, \boldsymbol{p}) - h(0_3, \boldsymbol{p})\big]\big|_{p_i=1/2} \quad i = 1, \ldots, 6 \\ &= \frac{7}{32}. \end{aligned}$$

Problem 1

Alternatively:

$$J_B^{(3)} = \text{the number of critical path sets for component } 3/(2^{6-1})$$

Since we have the following 7 critical path sets for component 3, the previous answer is verified:

$\{1,3,5\}, \{1,3,6\}, \{2,3,5\}, \{1,2,3,5\}, \{1,3,5,6\}, \{1,3,4,6\}, \{2,3,4,5\}.$

Problem 1

f) Assume that $p_i = p$ for $i = 1, 2, \ldots, 6$, i.e., that all the components have the same component reliability. What can you say about the reliability importance of the other 5 components?

SOLUTION: From symmetry in the reliability block diagram, we see that the reliability importance of components 1 and 5 must be the same. Also by symmetry, the reliability importance of components 2 and 6 must be the same. The explicit expressions for the reliability importance can be computed in a similar way as in e).

Problem 2

Consider a binary monotone system (C, ϕ), where $C = \{1, 2, 3\}$ and where the structure function ϕ is given by:

$$\phi(\boldsymbol{X}) = \mathrm{I}(\sum_{i=1}^{3} X_i \geq 2).$$

Here $\boldsymbol{X} = (X_1, X_2, X_3)$ denotes the vector of component state variables and $\mathrm{I}(\cdot)$ denotes the indicator function.

Problem 2

a) Show that the structure function ϕ can be written as:

$$\phi(\boldsymbol{X}) = X_1 X_2 + X_1 X_3 + X_2 X_3 - 2X_1 X_2 X_3.$$

SOLUTION: By using pivotal decomposition we get:

$$\begin{aligned}
\phi(\boldsymbol{X}) &= \mathrm{I}(\sum_{i=1}^{3} X_i \geq 2) \\
&= X_1 \phi(1_1, \boldsymbol{X}) + (1 - X_1)\phi(0_1, \boldsymbol{X}) \\
&= X_1 \cdot \mathrm{I}(\sum_{i=2}^{3} X_i \geq 1) + (1 - X_1) \cdot \mathrm{I}(\sum_{i=2}^{3} X_i \geq 2) \\
&= X_1 \cdot (X_2 \amalg X_3) + (1 - X_1) X_2 \cdot X_3 \\
&= X_1 \cdot (X_1 + X_3 - X_2 X_3) + (1 - X_1) X_2 \cdot X_3 \\
&= X_1 X_2 + X_1 X_3 + X_2 X_3 - 2X_1 X_2 X_3.
\end{aligned}$$

Problem 2

In the following we assume that:

$$X_i = Y_0 \cdot Y_i, \quad i = 1, 2, 3,$$

where Y_0, Y_1, Y_2, Y_3 are independent binary stochastic variables and:

$$P(Y_0 = 1) = \theta, \;\; P(Y_1 = 1) = P(Y_2 = 1) = P(Y_3 = 1) = q,$$

where $0 < \theta < 1$ and $0 < q < 1$.

Problem 2

b) Explain why this implies that X_1, X_2, X_3 are associated stochastic variables.

SOLUTION: Since X_1, X_2, X_3 are non-decreasing functions of the independent variables Y_0, Y_1, Y_2, Y_3, it follows that X_1, X_2, X_3 are associated stochastic variables.

Problem 2

We then introduce $h = \mathrm{E}[\phi(\boldsymbol{X})] = P(\phi(\boldsymbol{X}) = 1)$.

c) Show that:

$$h = h(\theta, q) = \theta q^2 (3 - 2q).$$

SOLUTION: By conditioning on the state of Y_0 we get:

$$\begin{aligned}
h &= h(\theta, q) = \mathrm{E}[\phi(\boldsymbol{X})] \\
&= P(Y_0 = 1) \cdot \mathrm{E}[\phi(\boldsymbol{X}) | Y_0 = 1] + P(Y_0 = 0) \cdot \mathrm{E}[\phi(\boldsymbol{X}) | Y_0 = 0] \\
&= \theta \cdot \mathrm{E}[Y_1 Y_2 + Y_1 Y_3 + Y_2 Y_3 - 2 Y_1 Y_2 Y_3] + (1 - \theta) \cdot 0 \\
&= \theta(3q^2 - 2q^3) = \theta q^2 (3 - 2q).
\end{aligned}$$

Problem 2

Assume that we ignore the dependence between the X_is, and instead compute the system reliability as if X_1, X_2, X_3 are independent and:

$$P(X_i = 1) = \theta q, \quad i = 1, 2, 3.$$

Let $\tilde{h}$ denote the system reliability we then get.

d) Show that:

$$\tilde{h} = \tilde{h}(\theta, q) = \theta^2 q^2 (3 - 2\theta q).$$

SOLUTION: Assuming that X_1, X_2, X_3 are independent we get:

$$\begin{aligned}
\tilde{h} &= \mathrm{E}[X_1 X_2 + X_1 X_3 + X_2 X_3 - 2X_1 X_2 X_3] \\
&= (\theta \cdot q)^2 + (\theta \cdot q)^2 + (\theta \cdot q)^2 - 2(\theta \cdot q)^3 \\
&= \theta^2 q^2 (3 - 2\theta q).
\end{aligned}$$

Problem 2

e) Assume that $\theta = \frac{1}{2}$. Show that we then have $\tilde{h} < h$ for all $0 < q < 1$.

SOLUTION: We introduce $d(\theta, q) = h(\theta, q) - \tilde{h}(\theta, q)$ given by:

$$\begin{aligned} d(\theta, q) = h(\theta, q) - \tilde{h}(\theta, q) &= \theta q^2(3 - 2q) - \theta^2 q^2(3 - 2\theta q) \\ &= \theta q^2[3 - 2q - 3\theta + 2\theta^2 q] \\ &= \theta q^2[3(1 - \theta) - 2q(1 - \theta^2)] \\ &= \theta(1 - \theta)q^2[3 - 2q(1 + \theta)] \end{aligned}$$

If $\theta = \frac{1}{2}$ we get for all $0 < q < 1$ that:

$$d(\theta, q) = (\frac{1}{4})q^2[3 - 2q(\frac{3}{2})] = (\frac{3}{4})q^2[1 - q] > 0.$$

Hence, $\tilde{h} < h$ for all $0 < q < 1$. Thus, in this case we always underestimate the system reliability if we ignore the dependence.

Problem 2

f) Assume instead that $\theta = \frac{3}{4}$. What can you say about the relationship between $\tilde{h}$ and h in this case?

SOLUTION: If $\theta = \frac{3}{4}$ we get that:

$$d(\theta, q) = (\frac{3}{16})q^2[3 - 2q(\frac{7}{4})] = (\frac{3}{32})q^2[6 - 7q]$$

Hence, $d(\theta, q) > 0$ if and only if $7q < 6$. Thus, we conclude that $\tilde{h} < h$ if and only if $q < \frac{6}{7}$. Conversely, $\tilde{h} > h$ if and only if $q > \frac{6}{7}$.
We observe that in this case we always underestimate the system reliability if we ignore the dependence when $q < \frac{6}{7}$, while we overestimate the system reliability if we ignore the dependence when $q > \frac{6}{7}$.

Problem 3

Let (C, ϕ) be a binary monotone system, and let $\boldsymbol{X}$ denote the vector of component state variables. In this problem we consider how the system reliability $h = P(\phi(\boldsymbol{X}) = 1)$ can be estimated using Monte Carlo simulation. The simplest Monte Carlo estimate is:

$$\hat{h}_{MC} = \frac{1}{N} \sum_{r=1}^{N} \phi(\boldsymbol{X}_r),$$

where $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ are data generated from the distribution of $\boldsymbol{X}$.

Problem 3

In order to improve this estimate we let $S = S(\boldsymbol{X})$ be a stochastic variable with values in the set $\{s_1, \ldots, s_k\}$. We assume that the distribution of S is known, and introduce:

$$\theta_j = E[\phi | S = s_j], \quad j = 1, \ldots, k.$$

We then use Monte Carlo simulation in order to estimate $\theta_1, \ldots, \theta_k$, and generate data from the conditional distribution of $\boldsymbol{X}$ given S. We let $\{\boldsymbol{X}_{r,j} : r = 1, \ldots, N_j\}$ denote the vectors generated from the distribution of $\boldsymbol{X}$ given that $S = s_j$, $j = 1, \ldots, k$, and get the following estimates:

$$\hat{\theta}_j = \frac{1}{N_j} \sum_{r=1}^{N_j} \phi(\boldsymbol{X}_{r,j}), \quad j = 1, \ldots, k.$$

These estimates are then combined into the following estimate of the system reliability:

$$\hat{h}_{CMC} = \sum_{j=1}^{k} \hat{\theta}_j P(S = s_j).$$

Problem 3

a) Show that $E[\hat{h}_{CMC}] = h$ and that the variance of the estimate is given by:

$$\text{Var}(\hat{h}_{CMC}) = \sum_{j=1}^{k} \frac{1}{N_j} \text{Var}(\phi | S = s_j)[P(S = s_j)]^2$$

SOLUTION: We first note that the variances of the estimates $\hat{\theta}_1, \ldots, \hat{\theta}_k$ are given by:

$$\text{Var}(\hat{\theta}_j) = \frac{1}{N_j^2} \sum_{r=1}^{N_j} \text{Var}(\phi | S = s_j) = \frac{1}{N_j} \text{Var}(\phi | S = s_j), \quad j = 1, \ldots, k.$$

Inserting this into the variance of $\hat{h}_{CMC}$ we get:

$$\text{Var}(\hat{h}_{CMC}) = \text{Var}[\sum_{j=1}^{k} \hat{\theta}_j P(S = s_j)] = \sum_{j=1}^{k} \text{Var}(\hat{\theta}_j)[P(S = s_j)]^2$$

Problem 3

b) Assume that $N_j \approx N \cdot P(S = s_j)$, $j = 1, \ldots, k$. Show that we then have:

$$\mathrm{Var}(\hat{h}_{CMC}) \approx \frac{1}{N}(\mathrm{Var}(\phi) - \mathrm{Var}[E(\phi|S)]),$$

and explain briefly why this implies that $\mathrm{Var}(\hat{h}_{CMC}) \leq \mathrm{Var}(\hat{h}_{MC})$.

Problem 3

SOLUTION: By inserting $N_j \approx N \cdot P(S = s_j)$, $j = 1, \ldots, k$ into the expression found in (a) we get:

$$\begin{aligned} Var(\hat{h}_{CMC}) &\approx \sum_{j=1}^{k} \frac{1}{N \cdot P(S = s_j)} \mathrm{Var}(\phi | S = s_j)[P(S = s_j)]^2 \\ &= \frac{1}{N} \sum_{j=1}^{k} \mathrm{Var}(\phi | S = s_j) P(S = s_j) = \frac{1}{N} E[\mathrm{Var}(\phi | S)] \\ &= \frac{1}{N} \big(\mathrm{Var}(\phi) - \mathrm{Var}[E(\phi | S)] \big), \end{aligned}$$

where we have used the formula:

$$\mathrm{Var}(\phi) = \mathrm{Var}[E(\phi | S)] + E[\mathrm{Var}(\phi | S)].$$

Problem 3

Since obviously $\mathrm{Var}[E(\phi|S)] \geq 0$, it follows that:

$$\frac{1}{N}\big(\mathrm{Var}(\phi) - \mathrm{Var}[E(\phi|S)]\big) \leq \frac{1}{N}\mathrm{Var}(\phi) = \mathrm{Var}(\hat{h}_{MC})$$

Hence, we conclude that $\mathrm{Var}(\hat{h}_{CMC}) \leq \mathrm{Var}(\hat{h}_{MC})$.

Problem 3

c) What should one take into account when choosing S?

SOLUTION: From (b) we see that the conditional estimate has smaller variance than the original Monte Carlo estimate provided that $\mathrm{Var}[E(\phi \mid S)]$ is positive. This quantity can be interpreted as a measure of how much information S contains relative to ϕ. Thus, when looking for good choices for S we should look for variables containing as much information about ϕ as possible. However, there are some other important points that need to be considered. First of all S must have a distribution that can be derived analytically in polynomial time. Secondly, the number of possible values of S, i.e., k, must be polynomially limited by n. Finally, it must be possible to sample efficiently from the distribution of $\boldsymbol{X}$ given S.

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