STK3405 - Exam 2019

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EXERCISE 1

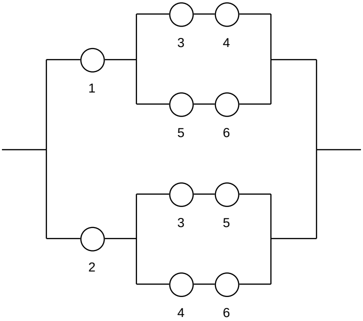

Figure: Block diagram of (C, ϕ)

a) Find the minimal path sets (4 sets) and the minimal cut sets (7 sets) of the system.

SOLUTION:

Minimal path sets:

$$P_1 = \{1,3,4\},\ P_2 = \{1,5,6\},\ P_3 = \{2,3,5\},\ P_4 = \{2,4,6\}.$$

Minimal cut sets:

$$K_1 = \{1,2\},\ K_2 = \{3,6\},\ K_3 = \{4,5\},\ K_4 = \{1,3,4\},$$
$$K_5 = \{1,5,6\},\ K_6 = \{2,3,5\},\ K_7 = \{2,4,6\}.$$

b) We let $h(\boldsymbol{p}) = P(\phi = 1)$ denote the reliability function of the system. Show that:

$$h(1_1, 1_2, \boldsymbol{p}) = E[\phi(1_1, 1_2, \boldsymbol{X})] = (p_3 \amalg p_6) \cdot (p_4 \amalg p_5),$$

$$h(1_1, 0_2, \boldsymbol{p}) = E[\phi(1_1, 0_2, \boldsymbol{X})] = (p_3 \cdot p_4) \amalg (p_5 \cdot p_6),$$

$$h(0_1, 1_2, \boldsymbol{p}) = E[\phi(0_1, 1_2, \boldsymbol{X})] = (p_3 \cdot p_5) \amalg (p_4 \cdot p_6),$$

and use this to find $h(\boldsymbol{p})$.

SOLUTION:

CASE 1. $X_1 = 1$ and $X_2 = 1$. In this case the minimal cut sets containing components 1 or 2 are functioning. Thus, the remaining minimal cut sets are $K_2 = \{3, 6\}$ and $K_3 = \{4, 5\}$. This implies that:

$$\phi(1_1, 1_2, \boldsymbol{X}) = (X_3 \amalg X_6) \cdot (X_4 \amalg X_5),$$

and hence:

$$h(1_1, 1_2, \boldsymbol{p}) = (p_3 \amalg p_6) \cdot (p_4 \amalg p_5).$$

CASE 2. $X_1 = 1$ and $X_2 = 0$. In this case the minimal paths containing component 2 have failed. Thus, we have a system with minimal path sets $P_1' = \{3, 4\}$ and $P_2' = \{5, 6\}$. This implies that:

$$\phi(1_1, 0_2, \boldsymbol{X}) = (X_3 \cdot X_4) \amalg (X_5 \cdot X_6),$$

and hence:

$$h(1_1, 0_2, \boldsymbol{p}) = (p_3 \cdot p_4) \amalg (p_5 \cdot p_6).$$

CASE 3. $X_1 = 0$ and $X_2 = 1$. In this case the minimal paths containing component 1 have failed. Thus, we have a system with minimal path sets $P_3' = \{3, 5\}$ and $P_4' = \{4, 6\}$. This implies that:

$$\phi(0_1, 1_2, \boldsymbol{X}) = (X_3 \cdot X_5) \amalg (X_4 \cdot X_6),$$

and hence:

$$h(0_1, 1_2, \boldsymbol{p}) = (p_3 \cdot p_5) \amalg (p_4 \cdot p_6).$$

By combining these results and the fact that $h(0_1, 0_2, \boldsymbol{p}) = \phi(0_1, 0_2, \boldsymbol{X}) = 0$, (since $\{1, 2\}$ is a minimal cut set) we get:

$$
\begin{aligned}
h(\boldsymbol{p}) &= p_1 p_2 h(1_1, 1_2, \boldsymbol{p}) + p_1(1 - p_2) h(1_1, 0_2, \boldsymbol{p}) + (1 - p_1) p_2 h(0_1, 1_2, \boldsymbol{p}) \\
&= p_1 p_2 [(p_3 \amalg p_6) \cdot (p_4 \amalg p_5)] + p_1(1 - p_2)[(p_3 \cdot p_4) \amalg (p_5 \cdot p_6)] \\
&+ (1 - p_1) p_2 [(p_3 \cdot p_5) \amalg (p_4 \cdot p_6)].
\end{aligned}
$$

In the remaining part of this problem we assume that all components have equal reliability p, i.e., $p_1 = \cdots = p_6 = p$. The reliability function can then be written as $h(p)$ instead of $h(\boldsymbol{p})$.

c) Use the results from (b) to show that:

$$h(p) = p^2 \cdot (2p - p^2)^2 + 2p(1-p) \cdot (2p^2 - p^4)$$
$$= p^4 \cdot (2-p)^2 + 2p^3(1-p)(2-p^2)$$

In particular, show that:

$$h(\frac{1}{2}) = 23 \cdot (\frac{1}{2})^6$$

SOLUTION:

We start by noting that:

$$s \amalg s = 1 - (1-s)(1-s) = 2s - s^2, \quad \text{for all } s. \tag{1}$$

By using (1) and inserting $p_1 = \cdots = p_6 = p$ into $h(\boldsymbol{p})$ we get:

$$h(p) = p^2[(2p - p^2) \cdot (2p - p^2)] + p(1-p)[2p^2 - p^4] + (1-p)p[2p^2 - p^4$$

$$= p^2 \cdot (2p - p^2)^2 + 2p(1-p) \cdot (2p^2 - p^4)$$

$$= p^4 \cdot (2-p)^2 + 2p^3(1-p)(2-p^2).$$

In particular we have:

$$h(\frac{1}{2}) = (\frac{1}{2})^4 \cdot (\frac{3}{2})^2 + 2 \cdot (\frac{1}{2})^3 \cdot (\frac{1}{2}) \cdot (\frac{7}{4})$$

$$= [3^2 + 2 \cdot 7] \cdot (\frac{1}{2})^6 = 23 \cdot (\frac{1}{2})^6.$$

d) Let $S = \sum_{i=1}^{6} X_i$. Explain why the distribution of S is given by:

$$P(S = s) = \binom{6}{s} p^s (1-p)^{6-s}, \quad s = 0, 1, \ldots, 6.$$

SOLUTION:

The random variable S is the sum of the independent and identically binary variables $X_1, \ldots, X_6$. Hence, $S \sim Bin(6, p)$.

e) Show that:

$$h(p) = \sum_{s=0}^{6} b_s p^s (1-p)^{6-s}$$

where b_s denotes the number of path sets (minimal and non-minimal) having exactly s components, $s = 0, 1, \ldots, 6$.

SOLUTION:

We start by noting that:

$$b_s = \sum_{\{\boldsymbol{X} : \sum_{i=1}^{6} x_i = s\}} \phi(\boldsymbol{x}), \quad s = 0, 1, \ldots, 6.$$

Moreover, the conditional distribution of $\boldsymbol{X}$ given S is:

$$P(\boldsymbol{X} = \boldsymbol{x} \mid S = s) = \frac{p^{\sum_{i=1}^{6} x_i}(1-p)^{6-\sum_{i=1}^{6} x_i}}{\binom{6}{s} p^s (1-p)^{6-s}} = \frac{1}{\binom{6}{s}},$$

for all $\boldsymbol{x}$ such that $\sum_{i=1}^{6} x_i = s$, and zero otherwise.

From this it follows that:

$$\begin{aligned} E[\phi(\boldsymbol{X})|S=s] &= \sum_{\{\boldsymbol{X} : \sum_{i=1}^{6} x_i = s\}} \phi(\boldsymbol{x}) P(\boldsymbol{X} = \boldsymbol{x} \mid S = s) \\ &= \frac{1}{\binom{6}{s}} \sum_{\{\boldsymbol{X} : \sum_{i=1}^{6} x_i = s\}} \phi(\boldsymbol{x}) \\ &= \frac{b_s}{\binom{6}{s}} , \end{aligned}$$

Finally, the system reliability, h, expressed as a function of p, is given by:

$$\begin{aligned} h(p) = E[\phi(\boldsymbol{X})] &= \sum_{s=0}^{6} E[\phi(\boldsymbol{X})|S=s] P(S=s) \\ &= \sum_{s=0}^{6} \frac{b_s}{\binom{6}{s}} \binom{6}{s} p^s (1-p)^{6-s} = \sum_{s=0}^{6} b_s p^s (1-p)^{6-s}. \end{aligned}$$

f) Show that:

$$\sum_{s=0}^{6} b_s = 23.$$

SOLUTION:

By inserting $p = \frac{1}{2}$ into the expression for $h(p)$ we get:

$$h(\frac{1}{2}) = \sum_{s=0}^{6} b_s(\frac{1}{2})^s(1-(\frac{1}{2}))^{6-s}$$

$$= \sum_{s=0}^{6} b_s(\frac{1}{2})^6 = [\sum_{s=0}^{6} b_s](\frac{1}{2})^6 = 23 \cdot (\frac{1}{2})^6,$$

where the last equality follows by the last result in (c). Hence, it follows that we must have:

$$\sum_{s=0}^{6} b_s = 23.$$

g) Finally, determine $b_0, b_1, \ldots, b_6$.

SOLUTION:

Since the smallest path sets have 3 components, we must have:

$$b_0 = b_1 = b_2 = 0.$$

We know from (a) that there are 4 minimal paths, all of size 3. Hence we have:

$$b_3 = 4.$$

Since all cut sets have at least 2 components, all sets of size 5 or 6 must be path sets. Hence, we have:

$$b_5 = \binom{6}{5} = 6, \text{ and } b_6 = \binom{6}{6} = 1.$$

In order to determine b_4, we could go through all sets of size 4, i.e., $\binom{6}{4} = 15$ sets, and count the path sets among these sets. Alternatively, we can apply the result from (f). This gives us an equation which we can use to determine b_4:

$$0 + 0 + 0 + 4 + b_4 + 6 + 1 = 23$$

which implies that:

$$b_4 = 12.$$

NOTE: By the same arguments as we have used in this problem we can show more generally that if (C, ϕ) is a binary monotone system of order n, then:

$$\sum_{s=0}^{n} b_s = h(\frac{1}{2}) \cdot 2^n.$$

EXERCISE 2

If $T_1, \ldots, T_n$ are random variables, and we let $\mathbf{T} = (T_1, \ldots, T_n)$, we say that $T_1, \ldots, T_n$ are *associated* if

$$\mathrm{Cov}(\Gamma(\mathbf{T}), \Delta(\mathbf{T})) \geq 0,$$

for all binary, non-decreasing functions Γ and Δ.

a) Prove that non-decreasing functions of associated random variables are associated.

SOLUTION:

Let $T_1, \ldots, T_n$ be associated, and let $\mathbf{T} = (T_1, \ldots, T_n)$. Moreover, we let $S_i = f_i(\mathbf{T})$, $i = 1, \ldots, m$, where $f_1, \ldots, f_m$ are non-decreasing functions, and let $\mathbf{S} = (S_1, \ldots, S_m)$. Finally, let $\Gamma = \Gamma(\mathbf{S})$ and $\Delta = \Delta(\mathbf{S})$ be binary non-decreasing functions. Then $\Gamma(\mathbf{S}) = \Gamma(f_1(\mathbf{T}), \ldots, f_m(\mathbf{T}))$ and $\Delta(\mathbf{S}) = \Delta(f_1(\mathbf{T}), \ldots, f_m(\mathbf{T}))$ are non-decreasing functions of $\mathbf{T}$ as well. Hence, by the definition of association, it follows that:

$$\mathrm{Cov}(\Gamma(\mathbf{S}), \Delta(\mathbf{S})) = \mathrm{Cov}(\Gamma(f_1(\mathbf{T}), \ldots, f_m(\mathbf{T})), \Delta(f_1(\mathbf{T}), \ldots, f_m(\mathbf{T}))) \geq 0.$$

Hence, we conclude that $S_1, \ldots, S_m$ are associated as well.

b) Assume that $T_1, \ldots, T_n$ are associated random variables such that $0 \leq T_i \leq 1$, $i = 1, \ldots, n$. Prove that

$$E[\prod_{i=1}^{n} T_i] \geq \prod_{i=1}^{n} E[T_i] \quad \text{and} \tag{2}$$

$$E[\coprod_{i=1}^{n} T_i] \leq \coprod_{i=1}^{n} E[T_i]. \tag{3}$$

SOLUTION:

We note that since $0 \leq T_i \leq 1$, both T_i and $S_i = 1 - T_i$ are non-negative random variables, $i = 1, \ldots, n$. Hence, the product functions $\prod_{i=1}^n T_i$ and $\prod_{i=1}^n S_i$ are both non-decreasing in each argument.

Since non-decreasing functions of associated random variables have non-negative covariance, we find:

$$E[\prod_{i=1}^n T_i] - E[T_1]E[\prod_{i=2}^n T_i] = \mathrm{Cov}(T_1, \prod_{i=2}^n T_i) \geq 0,$$

since the product function is non-decreasing in each argument because $T_i \geq 0$, $i = 2, \ldots, n$. This implies that:

$$E[\prod_{i=1}^n T_i] \geq E[T_1]E[\prod_{i=2}^n T_i].$$

By repeated use of this inequality, we get (1).

Similarly to a), we can prove that non-increasing functions of associated random variables are associated. By using this, $S_1, \ldots, S_n$ are associated random variables. Moreover, $0 \leq S_i \leq 1$, $i = 1, \ldots, n$, so we can apply (1) to these variables. From this it follows that:

$$
\begin{aligned}
E[\prod_{i=1}^{n} T_i] &= 1 - E[\prod_{i=1}^{n}(1 - T_i)] = 1 - E[\prod_{i=1}^{n} S_i] \\
&\leq 1 - \prod_{i=1}^{n} E(S_i) = 1 - \prod_{i=1}^{n}(1 - E[T_i]) \\
&= \prod_{i=1}^{n} E[T_i],
\end{aligned}
$$

so (2) is proved as well.

c) Interpret the inequalities (2) and (3) by applying them to the binary component state variables $X_1, \ldots, X_n$.

SOLUTION:

If we apply the result in b) to the binary component state variables $X_1, \ldots, X_n$, inequality (2) says that for a series structure of associated components, an incorrect assumption of independence will lead to an *underestimation* of the system reliability. Correspondingly, inequality (3) says that for a parallel structure, an incorrect assumption of independence between the components will lead to an *overestimation* of the system reliability. Clearly, overestimating the system reliability can have serious consequences in applications, and should be avoided. Since most systems are not purely series or purely parallel, we conclude that for an arbitrary structure, we *cannot say for certain what the consequences of an incorrect assumption of independence will be*. This means that in any application where we do not know for sure that the components are independent, simply assuming independence in order to compute the exact system reliability can be dangerous.

d) Let $X_1, \ldots, X_n$ be the associated component states of a binary monotone system (C, ϕ) with minimal path series structures $(P_1, \rho_1), \ldots, (P_p, \rho_p))$ and minimal cut parallel $(K_1, \kappa_1), \ldots, (K_k, \kappa_k))$. Prove that

$$\prod_{j=1}^{k} P(\kappa_j(\mathbf{X}^{K_j}) = 1) \leq h \leq \coprod_{j=1}^{p} P(\rho_j(\mathbf{X}^{P_j}) = 1). \tag{4}$$

Hint: Use the results from items a) and b).

SOLUTION:

Since non-decreasing functions of associated random variables are associated from a), it follows that the minimal path series structures, and the minimal cut parallel structures, are associated. Hence, we get:

$$\begin{aligned}
\prod_{j=1}^{k} P(\kappa_j(\mathbf{X}^{K_j}) = 1) &\leq E[\prod_{j=1}^{k} \kappa_j(\mathbf{X}^{K_j}] \\
&= h \\
&= E[\coprod_{j=1}^{p} \rho_j(\mathbf{X}^{P_j})] \\
&\leq \coprod_{j=1}^{p} P(\rho_j(\mathbf{X}^{P_j}) = 1),
\end{aligned}$$

the first inequality follows from inequality (2), the first and second equalities follow by representing the system via its minimal path series and cut parallel structures. The final inequality follows from (3).

e) Make the same assumptions as in item d), and assume in addition that the component states are independent with component reliabilities $p_1, p_2, \ldots p_n$. Use the result in d) to prove that

$$\prod_{j=1}^{k} \coprod_{i \in K_j} p_i \leq h(\mathbf{p}) \leq \coprod_{j=1}^{p} \prod_{i \in P_j} p_i. \tag{5}$$

SOLUTION:

For independent components,

$$P(\kappa_j(\mathbf{X}^{K_j}) = 1) = E[\coprod_{i \in K_j} X_i] = \coprod_{i \in K_j} p_i.$$

Hence, the lower bound follows from the result in d). The upper bound is proved in the same way.

f) Consider the system in Problem 1. Assume that all components have the same component reliability $p = 0.9$. Compute the bounds in inequality (5) and comment on how well they approximate the actual system reliability in this case.

SOLUTION:

Insert $p = 0.9$ in the bounds from item e) as well as in the expression for $h(\mathbf{p})$ from Problem 1.
By considering the minimal cut sets found in Problem 1, we observe that the first 3 sets are of size 2, while the 4 last sets are of size 3. In order to compute the lower bound, we note that by (1) we have:

$$\coprod_{i \in K_1} p = \coprod_{i \in K_2} p = \coprod_{i \in K_3} p = p \amalg p = (2p - p^2)$$

Moreover, similar calculations yield that:

$$\coprod_{i \in K_4} p = \coprod_{i \in K_5} p = \coprod_{i \in K_6} p = \coprod_{i \in K_7} p = p \amalg p \amalg p = (3p - 3p^2 + p^3)$$

Hence, the lower bound becomes:

$$\ell(p) = \prod_{j=1}^{7} \coprod_{i \in K_j} p = (2p - p^2)^3 \cdot (3p - 3p^2 + p^3)^4$$

By considering the minimal path sets found in Problem 1, we observe that all 4 sets are of size 3. Hence, upper bound becomes:

$$u(p) = \coprod_{j=1}^{4} \prod_{i \in P_j} p = \coprod_{j=1}^{4} p^3 = 1 - (1 - p^3)^4.$$

By inserting $p = 0.9$ we get:

$$\begin{aligned} \ell(0.9) &= 0.9664, \\ h(0.9) &= 0.9674, \\ u(0.9) &= 0.9946. \end{aligned}$$

We observe that the lower bound is very close to the correct system reliability, while the upper bound is noticeably higher.

g) In points d) and e), you have found upper and lower bounds for the system reliability. In which cases is it particularly important to have such bounds?

SOLUTION:

It is often the case that we are unable to compute the exact reliability of a system. This may be the case for large, complex systems where the computations simply take too much time, but also for systems where the components are not independent. Bounds for the system reliability can be useful whenever computing the exact reliability is impossible or too computationally costly.