Problems and Methods in Actuarial Science (STK 3505)
Exercises 2, 16.9.2015

Problem 1 Show that

$$ {}_kp_x = p_x p_{x+1} p_{x+2} \cdot \ldots \cdot p_{x+k-1}, \; k = 1, 2, 3, \ldots, $$

where ${}_kp_x$ and p_{x+j} are (conditional) survival probabilities of a life (x) aged x years.

Problem 2
(i) Verify the relations

$$ {}_{s+t}p_x = {}_sp_x \cdot {}_t p_{x+s} $$

and

$$ {}_{s|t}q_x = {}_sp_x \cdot {}_t q_{x+s}. $$

(ii) Let μ_{x+t} be the force of mortality of a life (x) at the age $x+t$. Argue why

$$ {}_sq_{x+t} \approx \mu_{x+t} \cdot s $$

for small values of s.

Problem 3 Assume the polynomial growth model of Weibull (1939) for the dynamics of the force of mortality, that is μ_{x+t} is described by

$$ \mu_{x+t} = k(x+t)^n $$

for fixed parameters $k, n > 0$.

Derive an explicit formula for the survival probability ${}_tp_x$.

Problem 4 Consider an endowment for which a payment of 1 unit is provided at the end of the year of death, if this occurs within the first n years, otherwise at the end of the nth year. Thus its present value Z is given by

$$ Z = Z_1 + Z_2 $$

with

$$ Z_1 = v^{K+1} 1_{\{K<n\}} $$

is the present value of a term insurance and

$$ Z_2 = v^n 1_{\{K \geq n\}} $$

is the present value of a pure endowment. Show that the variance of Z can be written as

$$ Var[Z] = Var[Z_1] + Var[Z_2] - 2A^1_{x:\overline{n}|} \cdot A_{\substack{1 \\ x:\overline{n}|}}, \tag{1} $$

where $A^1_{x:\overline{n}|} = E[Z_1]$ and $A_{\substack{1 \\ x:\overline{n}|}} = E[Z_2]$ are the corresponding net single premiums.

Give an interpretation of relation (1).

Problem 5 Consider the following table of values of e_x :

Age x	e_x
75	10.5
76	10.0
77	9.5

Compute the probability that a life age 75 will survive to age 77.

Hint: Prove that

$$e_{x+j} = p_{x+j}(1 + e_{x+j+1}),\ \ j = 0, 1,$$

where

$$e_{x+j} = \sum_{k \geq 1} {}_k p_{x+j},\ \ j = 0, 1.$$

Problem 6 Denote by K the curtate of the future life time $T = T(x)$ of a life (x) and let S be the remaining fraction of the year during which (x) is alive in the year of death, i.e.

$$S = T - K.$$

(i) Show that the random variable S has a continuous distribution between 0 and 1.

(ii) Assume that the random variable S and K are independent and that S is uniformly distributed on $[0, 1]$, i.e.

$$\Pr(S \leq t) = t,\ 0 \leq t \leq 1.$$

Prove that

$$Var[T] = Var[K] + \frac{1}{12},$$

where $Var[X]$ denotes the variance of a random variable X, i.e.

$$Var[X] := E\left[(X - E[X])^2\right].$$

Problem 7 Assume that the sum insured is payable at the end of the mth part of the year of death, that is at time $K + S^{(m)}$, where K is the curtate of the future life time T and $S^{(m)} = \frac{1}{m}[mS + 1]$ with $S = T - K$ and $[\cdot]$ the Gauss bracket. So the present value of a whole life insurance of 1 unit becomes

$$Z = v^{K+S^{(m)}}.$$

Denote by $A_x^{(m)}$ its net single premium.

Suppose that K, S are independent random variables and that S is uniformly distributed on $[0, 1]$. Verify the relation

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x,$$

where A_x is the net single premium of an ordinary whole life insurance.

Problem 8 Let $c(t)$, $t \geq 0$ be a function. Consider an insurance with a payment of $c(T)$ at the instant of death. Thus its present value is given by

$$Z = c(T)v^T.$$

Then its net single premium has the representation

$$E[Z] = \sum_{j \geq 0} l_{j+1} \cdot v^{j+1} \cdot {}_j\, p_x \cdot q_{x+j},$$

where

$$l_{j+1} = E[c(j+S)(1+i)^{1-S} \mid K = j].$$

See e.g. Section 3 in Gerber.

Let us impose on the force of mortatlity μ_{x+t} the extrapolation condition

$$\mu_{x+j+u} = \mu_{x+j+\frac{1}{2}} \text{ for } 0 < u < 1,\ j \in \mathbb{N}_0.$$

Compute l_{j+1}.

For the choice of an exponentially increasing sum insured, that is $c(t) = e^{\tau t}, \tau \geq 0$, derive the formula

$$l_{j+1} = e^{\tau j} \frac{\mu_{x+j+\frac{1}{2}}}{1 - p_{x+j}} \frac{e^{\delta} - p_{x+j} e^{\tau}}{\delta + \mu_{x+j+\frac{1}{2}} - \tau},$$

provided the denominator is unequal zero. Here δ is the force of interest.