① Exercises 1

Problem 1 perpetuity (i.e. payment stream of infinite duration) with parameters

m # of payments per year

q # of payment increases

$\longrightarrow$ cash flow

time	paym.
$0, \frac{1}{m}, \dots, \frac{1}{q}-\frac{1}{m}$	$\frac{1}{m\cdot q}$
$\frac{1}{q}, \frac{1}{q}+\frac{1}{m}, \dots, \frac{2}{q}-\frac{1}{m}$	$\frac{1}{m\cdot q}+\frac{1}{m\cdot q}$
$\frac{2}{q}, \frac{2}{q}+\frac{1}{m}, \dots, \frac{3}{q}-\frac{1}{m}$	$\frac{1}{m\cdot q}+\frac{1}{m\cdot q}+\frac{1}{m\cdot q}$
$\vdots$	$\vdots$

can be = decomposed into the foll. sum of cashflows

time	paym
0 …	$\frac{1}{m\cdot q}$
$\frac{1}{q}$ …	$\frac{1}{m\cdot q}$
$\frac{2}{q}$ …	$\frac{1}{m\cdot q}$
$\vdots$	$\vdots$

perp. starting in 0

$+$

time	paym
0 …	0
$\frac{1}{q}$ …	$\frac{1}{m\cdot q}$
$\frac{2}{q}$ …	$\frac{1}{m\cdot q}$
$\vdots$	$\vdots$

perp. start. in $\frac{1}{q}$

$+$

time	paym
0 …	0
$\frac{1}{q}$ …	0
$\frac{2}{q}$ …	$\frac{1}{m\cdot q}$
$\vdots$	$\vdots$

perp. start. in $\frac{2}{q}$

$+ \dots$

$\longrightarrow$ perp. with constant payments $\frac{1}{m\cdot q}$

$$\Rightarrow \left(I^{(q)}\ddot{a}\right)^{(m)}_{\overline{\infty}|} = \sum_{l\geq 0} \underbrace{\sum_{k\geq 0} \left(\frac{1}{m\cdot q}\right)\left(v^{\frac{1}{q}}\right)^{l}\cdot\left(v^{\frac{1}{m}}\right)^{k}}_{\text{perp. starting in } \frac{l}{q} \text{ with paym. } \frac{1}{m\cdot q}}$$

$$= \left(\sum_{l\geq 0}\frac{1}{q}\left(v^{\frac{1}{q}}\right)^{l}\right)\left(\sum_{k\geq 0}\frac{1}{m}\left(v^{\frac{1}{m}}\right)^{k}\right)$$

$$= \ddot{a}^{(m)}_{\overline{\infty}|}\cdot\ddot{a}^{(q)}_{\overline{\infty}|} \overset{\text{Ex 2.5.3}}{=} \frac{1}{d^{(m)}}\frac{1}{d^{(q)}}$$

$d^{(l)}$ nom. discount rate with conv. period $\frac{1}{l}$

Ex. 1

(2) Problem 2

(i) a) $a_{\overline{n}|} = \sum_{k=1}^{n} v^k = \frac{1-v^{n+1}}{1-v} - 1 \cdot \frac{1-v}{1-v}$

$= \frac{1-v^{n+1}-1+v}{1-v} = \frac{v-v^{n+1}}{1-v} =$

$\frac{v(1-v^n)}{1-v} = \frac{1}{i}(1-v^n)$, since $v = \frac{1}{1+i}$

b) $\ddot{a}_{\overline{n}|}^{(m)} = \sum_{k=0}^{n \cdot m - 1} \frac{1}{m}\left(v^{\frac{1}{m}}\right)^k =$

$\frac{1}{m} \frac{1-\left(v^{\frac{1}{m}}\right)^{n \cdot m}}{1-v^{\frac{1}{m}}} = \left(\underbrace{\frac{1}{m} \cdot \frac{1}{1-v^{\frac{1}{m}}}}_{=\frac{1}{d^{(m)}} \text{ (Ex. 2.5.3)}}\right)(1-v^n)$

c) $a_{\overline{n}|}^{(m)} = \sum_{k=1}^{n \cdot m} \frac{1}{m}\left(v^{\frac{1}{m}}\right)^k = \frac{1}{m} v^{\frac{1}{m}} \left(\sum_{l=0}^{n \cdot m - 1} \left(v^{\frac{1}{m}}\right)^l\right)$

$= \frac{1}{m} v^{\frac{1}{m}} \frac{(1-v^n)}{(1-v^{\frac{1}{m}})} = \underbrace{\frac{1}{m} \cdot \frac{v^{\frac{1}{m}}}{(1-v^{\frac{1}{m}})}}_{=\frac{1}{i^{(m)}} \text{ (Ex. 2.5.4)}} (1-v^n)$

(ii) $s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$

$\Rightarrow \frac{1}{s_{\overline{n}|}} + \frac{1}{i} = \frac{i}{(1+i)^n - 1} + i = \frac{i(1+i)^n}{(1+i)^n - 1}$

$= i \cdot \frac{1}{1-v^n} = \frac{1}{a_{\overline{n}|}}$

Problem 3 :

S_k principal outstanding, i.e. value of remaining debt at $t=k$ without r_k!

$\rightarrow$ payments 0 at $t=k$, r_{k+1} at $t=k+1$, ..., r_n at $t=n$

$\Rightarrow S_k = v r_{k+1} + v^2 r_{k+2} + \dots + v^{n-k} r_n$ (prospective formula for S_k) (*)

$S := \sum_{h=1}^{n} v^h r_k$ (PV of the debt)

$\overset{(*),}{\underset{v=\frac{1}{1+i}}{\Longrightarrow}}$ $S_k = (1+i)^k S - \sum_{h=1}^{k} (1+i)^{k-h} r_h$ (**)

(retrospective formula for S_k; looking at the prev. paym.)

Ex 1

③ Use (2.1.2):

$$F_n = (1+i)^n F_0 + \sum_{k=1}^{n} (1+i)^{n-k} r_k$$

(solves the recursion

$$r_k = i(-F_{k-1}) + (F_k - F_{k-1}) \quad \text{(see (2.1.1))}$$

Choose $F_k = -S_k \implies$ proof

$$r_k = \underbrace{i S_{k-1}}_{\text{int. of running debt}} + \underbrace{(S_{k-1} - S_k)}_{\text{red. of principal}}$$

Problem 4:

bondholder / or lender gets the foll. payment stream from the bond issuer / or borrower

F face value

c coupon rate

C redemption value

$$\longrightarrow \quad PV = c \cdot F \underbrace{\frac{1-(1+i)^{-N}}{i}}_{= a_{\overline{N|}}} + C(1+i)^{-N}$$

i yield to maturity

Company wants to retire the bond

$\Rightarrow$ has to pay the value of the bond at Jan. 1, 2015

$$\rightarrow B = \sum_{k=1}^{5} 15000 \left(\frac{1}{1+0.06}\right)^k = 15000\, a_{\overline{5|}}$$

$i = 0.06$, $C = 0$

For this purpose the company deposits money

$\rightarrow$ value of the deposits (Jan. 1, 2015)

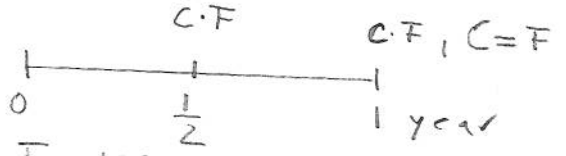
$$D = \sum_{k=0}^{9} x (1+0.06)^k = x \left(\frac{1-(1+0.06)^{10}}{1-(1+0.06)}\right) = x \cdot s_{\overline{10|}}$$

$\rightarrow$ Objective: $B = D$

$$\Rightarrow x = 15000 \frac{a_{\overline{5|}}}{s_{\overline{10|}}} = 4794$$

Problem 5:

bond of value 98.51 has cash flow



$F = 100,\quad c = 0.04/2$

$$\Rightarrow \quad 98.51 = PV = \underbrace{c\cdot F}_{=2} \cdot \frac{1}{1+\frac{i^{(2)}}{2}} + \underbrace{c\cdot F}_{=2} \cdot \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^2} + \underset{\overset{\parallel}{F}}{C} \cdot \frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^2}$$

$$= 2\,\overbrace{\frac{1}{1+\frac{i^{(2)}}{2}}}^{=x} + 102\,\frac{1}{\left(1+\frac{i^{(2)}}{2}\right)^2}$$

$$\rightarrow \quad x = 0.9729882 \qquad \Rightarrow \quad i^{(2)} = 5.55\,\%$$