① Exercises 2

Problem 1:

Recall that

$$ {}_tp_{x+s} \overset{def.}{=} \Pr(T > s+t \mid T > s) = \frac{{}_{t+s}p_x}{{}_sp_x} $$

prob. (x) survives another t years, after having reached x+s years

$$ {}_up_x = 1 - G(u) = \Pr(T > u) $$

$$ \Rightarrow \quad p_x \cdot p_{x+1} \cdot p_{x+2} \cdots p_{x+k-1} $$

$$ = {}_1p_x \cdot \frac{\cancel{{}_2p_x}}{\cancel{{}_1p_x}} \cdot \frac{\cancel{{}_3p_x}}{\cancel{{}_2p_x}} \cdots \frac{\cancel{{}_{k-1}p_x}}{{}_{k-2}p_x} \cdot \frac{{}_kp_x}{\cancel{{}_{k-1}p_x}} = {}_kp_x $$

Problem 2:

(i) a) $$ {}_tp_{x+s} \overset{\text{Prob. 1}}{=} \frac{{}_{t+s}p_x}{{}_sp_x} \Longrightarrow {}_{t+s}p_x = {}_sp_x \cdot {}_tp_{x+s} $$

b) $$ {}_{s|t}q_x = G(s+t) - G(s) = \underbrace{[1 - G(s)]}_{= {}_sp_x} \underbrace{\frac{G(s+t) - G(s)}{1 - G(s)}}_{= \Pr(T \le s+t \mid T > s) \overset{def}{=} {}_tq_{x+s}} $$

(ii)

$$ {}_sq_{x+t} = \frac{G(s+t) - G(t)}{1 - G(t)} $$

$$ = \left(\frac{\overbrace{\frac{G(t+s) - G(t)}{s}}^{\approx G'(t) = g(t) \text{ for } s \text{ small}}}{1 - G(t)} \right) \cdot s $$

$$ \approx \underbrace{\frac{g(t)}{1 - G(t)}}_{= \mu_{x+t}} \cdot s = \mu_{x+t} \cdot s $$

Problem 3:

$$ {}_tp_x \overset{(3.2.2)}{=} e^{-\int_0^t \mu_{x+s}\, ds} $$

$$ = e^{-\int_0^t K(x+t)^n\, ds} $$

$$ = \exp\left(-\frac{K}{n+1} (x+s)^{n+1} \Big|_{s=0}^{t} \right) $$

$$ = \exp\left(\frac{-K}{n+1} \left[(x+t)^{n+1} - x^{n+1} \right] \right) $$

Problem 6

$$ K \overset{def.}{=} [T(x)], \quad S = T - K \in [0, 1) $$

(i)

② Let $t \in [0,1]$

$$\Rightarrow \Pr(S < t) = \Pr((T - \overbrace{[T]}^{\overset{\text{def.}}{=} K}) < t)$$

$$= \Pr\Big(\bigcup_{l \geq 0} \underbrace{\{ l \leq T < l+t \}}_{=: A_l}\Big)$$

$$\underset{i \neq j}{\overset{A_i \cap A_j = \emptyset,}{=}} \sum_{l \geq 0} \Pr(l \leq T < l+t)$$

$$= \sum_{l \geq 0} (G(l+t) - G(l))$$

$$= \sum_{l \geq 0} \int_l^{l+t} g(s)\, ds = \sum_{l \geq 0} \int_0^t g(l+s)\, ds$$

$$\overset{\text{mon. convergence}}{=} \int_0^t \underbrace{\Big(\sum_{l \geq 0} g(l+s)\Big)}_{=: f \text{ density}} ds$$

(ii) $\mathrm{Var}[T] \overset{\text{def}}{=} \mathrm{Var}[K+S]$

$$= E[(K+S - E[K+S])^2]$$

$$= E[(\underbrace{K - E[K]}_{=: A} + \underbrace{S - E[S]}_{=: B})^2]$$

$$= E[A^2 + 2A \cdot B + B^2]$$

$$= \underbrace{E[A^2]}_{= \mathrm{Var}[K]} + 2E[A \cdot B] + \underbrace{E[B^2]}_{= \mathrm{Var}[S]} \qquad (*)$$

K, S indep. $\Rightarrow A, B$ indep. r.v.'s

$$\Rightarrow E[A \cdot B] = E[A] \cdot E[B]$$

Recall:
X, Y indep. r.v.'s, i.e.
$\Pr(X \leq x, Y \leq y) = \Pr(X \leq x) \cdot \Pr(Y \leq y)$
$\Rightarrow E[X \cdot Y] = E[X]\, E[Y]$

$$\Rightarrow (*) = \mathrm{Var}[K] + 2 \underbrace{E[A]}_{=0} \cdot \underbrace{E[B]}_{=0} + \mathrm{Var}[S]$$

$$= \mathrm{Var}[K] + \mathrm{Var}[S] \qquad (**)$$

③ Ex. 2

$$Var[S] = E[(S-E[S])^2]$$

$g(u) \equiv 1$ density of $Pr(S \le u)$

$$\Rightarrow E[S] \overset{(3.1.5)}{=} \int_0^1 u \cdot 1 \, du = \frac{1}{2}$$

$$\Rightarrow E[(S-\underbrace{E[S]}_{=\frac{1}{2}})^2] = \int_0^1 (u-\tfrac{1}{2})^2 du$$

$$= \frac{1}{3}(u-\tfrac{1}{2})^3 \Big|_{u=0}^{1} = \frac{1}{12}$$

$$\Rightarrow (**) = Var[K] + \frac{1}{12}$$

Problem 5:

age x	e_x
75	10.5
76	10.0
77	9.5

$_2p_{75}$?

$$_2p_{75} \overset{\text{Prob. 1}}{=} p_{75} \cdot \overbrace{p_{75+1}}^{\text{cond. prob.}}$$

Hint $\rightarrow$ $p_{75} = \dfrac{e_{75}}{1+e_{75+1}} = \dfrac{10.5}{1+10.0}$

$p_{75+1} = \dfrac{e_{75+1}}{1+e_{75+2}} = \dfrac{10.0}{1+9.5}$

$$\Rightarrow \quad _2p_{75} = 0.909$$

Remains to show: $e_{x+j+1} = \dfrac{e_{x+j}}{p_{x+j}} - 1$

W.l.o.g. $j=0$

$$\frac{e_x}{p_x} = \sum_{k\ge 1} \frac{_kp_x}{p_x} = \overbrace{\frac{_1p_x}{_1p_x}}^{=1} + \sum_{k \ge 2} \frac{_kp_x}{p_x}$$

$$= 1 + \underbrace{\sum_{\ell \ge 1} \frac{_{\ell+1}p_x}{p_x}}_{\overset{def}{=} e_{x+1}} \qquad \frac{_{\ell+1}p_x}{p_x} \overset{def}{=} {}_\ell p_{x+1}$$

$$= 1 + e_{x+1}$$

Ex. 2

Prob. 4 : Consider an endowment (i.e. means a mixture of a whole life, term ins. or pure endowment) with PV :

$$Z = Z_1 + Z_2,$$

where

$$Z_1 = v^{K+1} 1_{\{K<n\}}$$

PV of a term insur.

and

$$Z_2 = v^n 1_{\{K \geq n\}}$$

PV of a pure endowment

Observe that

$$Z_1 \cdot Z_2 = 0$$

$$\Rightarrow \quad Var[Z] = E[(Z_1 - E[Z_1] + Z_2 - E[Z_2])^2]$$

$$= Var[Z_1] + Var[Z_2] + \underbrace{E[2Z_1Z_2]}_{=0} - 2\underbrace{E[Z_1]}_{=A^1_{x:\overline{n}|}}\underbrace{E[Z_2]}_{=A_{x:\overline{n}|}^{\ \ 1}}$$

Var[Z] risk measure ; measures the risk of the financial position given by the pay off profile of

→ interpret :

$$\underbrace{Var[Z_1] + Var[Z_2]}_{\text{risk of selling the two insurances separately}} \geq \underbrace{Var[Z]}_{\text{risk of selling these policies in one endowment}}$$

Prob. 7

$$Z = v^{K+S^{(m)}}$$

PV of a payment of 1 at the end of the mth. part of the remaining life time in the year of death

$$S^{(m)} = \frac{1}{m}[mS+1]$$

e.g. $S = \frac{2}{m} + \frac{1}{2m} \rightarrow S^{(m)} = \frac{3}{m}$

; rounds to the next upper multiple of $\frac{1}{m}$

Ass. : K, S indep , S unif. distr.

$A_x^{(m)}$?

$$A_x^{(m)} = E[v^{K+S^{(m)}}] = E[v^{K+1} v^{-(1-S^{(m)})}]$$

$$\underset{K, S^{(m)} \text{ indep.}}{=} \underbrace{E[v^{K+1}]}_{=A_x} E[v^{-(1-S^{(m)})}] \qquad (*)$$

$$E[v^{-(1-S^{(m)})}] \underset{v=\frac{1}{1+i}}{=} \int_0^1 (1+i)^{1-\frac{1}{m}[m\cdot u+1]} du$$

$$= \sum_{l=1}^{m} \int_{[\frac{l-1}{m}, \frac{l}{m})} (1+i)^{1-\underbrace{\frac{1}{m}[m\cdot u+1]}_{=\frac{l}{m}}} du$$

$$= \sum_{\ell=1}^{m} (1+i)\,(1+i)^{-\frac{\ell}{m}} \cdot \underbrace{\int_{(\frac{\ell-1}{m}, \frac{\ell}{m}]} 1\,du}_{=\frac{1}{m}}$$

$$= (1+i) \underbrace{\left(\sum_{\ell=1}^{m} \frac{1}{m} v^{\frac{\ell}{m}} \right)}_{= a_{\overline{1|}}^{(m)} \overset{\text{Ex. 2}}{=} \frac{1-v}{i^{(m)}}} = \underbrace{(1+i)(1-v)}_{=i} \cdot \frac{1}{i^{(m)}} = \frac{i}{i^{(m)}}$$

Prob. 8 :

$$Z = c(T) \cdot v^{T}$$

PV of a paym. of $c(T)$ at the instant of death

Ass.: $\mu_{x+j+u} = \mu_{x+j+\frac{1}{2}}, \; 0 < u < 1, \; j \in \mathbb{N}_0$

$$E[Z] = \sum_{j \geq 0} \ell_{j+1}\, v^{j+1}\, {}_{j}p_x \cdot q_{x+j} \quad \text{with}$$

$$\ell_{j+1} = E[\, c(j+S)\,(1+i)^{1-S} \mid K = j \,]$$

ℓ_{j+1} ?

$$\Pr(S \leq u \mid K=j) = \frac{\overbrace{\Pr(j \leq T < j+u)}^{\overset{\text{def.}}{=} {}_{j|u}q}}{\underbrace{\Pr(K=j)}_{= {}_{j}p_x \cdot q_{x+j}}} = \frac{{}_{j|u}q}{{}_{j}p_x \cdot q_{x+j}} \overset{(*)}{\underset{\text{Ex 2, Pr. 2}}{=}} \frac{{}_{j}p_x \cdot {}_{u}q_{x+j}}{{}_{j}p_x \cdot q_{x+j}}$$

$$= \frac{{}_{u}q_{x+j}}{q_{x+j}}$$

$${}_{u}p_{x+j} \overset{(3.2.2)}{=} e^{-\int_0^u \overset{\mu_{x+j+\frac{1}{2}}}{\overline{\mu_{x+j+s}}}\, ds} = e^{-u\, \mu_{x+j+\frac{1}{2}}}$$

$$\Longrightarrow \frac{d}{du} \Pr(S \leq u \mid K=j) = \frac{d}{du} \left(\frac{1 - e^{-u \mu_{x+j+\frac{1}{2}}}}{q_{x+j}} \right)$$

$$= \mu_{x+j+\frac{1}{2}} \frac{e^{-u \mu_{x+j+\frac{1}{2}}}}{q_{x+j}} = \mu_{x+j+\frac{1}{2}} \frac{\Big(\overbrace{e^{-\mu_{x+j+\frac{1}{2}}}}^{\overset{\text{ass.}}{=} p_{x+j}} \Big)^{u}}{\underbrace{q_{x+j}}_{= 1 - p_{x+j}}}$$

$$= \mu_{x+j+\frac{1}{2}} \frac{(p_{x+j})^u}{1 - p_{x+j}} \quad \text{cond. density}$$

$$\overset{(*)}{\underset{\text{cond. dens.}}{\Longrightarrow}} \ell_{j+1} = \int_0^1 c(j+u)\,(1+i)^{1-u} \left(\mu_{x+j+\frac{1}{2}} \frac{(p_{x+j})^u}{1-p_{x+j}} \right) du$$

$$c(t) = e^{\tau t} \Longrightarrow c(j+u) = e^{\tau (j+u)}$$

$$(1+i) = e^{\delta} \Longrightarrow (1+i)^{1-u} = e^{\delta(1-u)}$$

$$(p_{x+j})^u = e^{u \log p_{x+j}}$$

Ex. 2

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$$\Rightarrow l_{j+1} = \frac{\mu_{x+j+\frac{1}{2}} e^{\tilde{i} \cdot j + \delta}}{1 - p_{x+j}} \int_0^1 e^{u(\tilde{i} - \delta + \log p_{x+j})} \, du$$

$$= \frac{\mu_{x+j+\frac{1}{2}} \cdot e^{\tilde{i} \cdot j + \delta}}{1 - p_{x+j}} \cdot \frac{e^{\tilde{i} - \delta} \cdot p_{x+j} - 1}{\tilde{i} - \delta + \underbrace{\log p_{x+j}}_{= -\mu_{x+j+\frac{1}{2}}}}$$