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Exercises 3

Prob. 1: $Z = c(K)\, v^{K+1}\, 1_{\{K<3\}}$

$\longrightarrow$ PV of a paym. of $c(K)$ at $t = K+1$, if $K < 3$

$\Pi := E[Z]$

$Pr(Z > \Pi)$?

$$\Pi = E[Z] = \sum_{j=0}^{2} c(j)\, v^{j+1}\, {}_jP_x \cdot q_{x+j}$$

$${}_1P_x = 1 - q_x = 1 - q_{x+0} = 0.8$$

$${}_2P_x = P_x \cdot \underbrace{P_{x+1}}_{1-q_{x+1}} = 0.8 \cdot 0.75$$

$$\Rightarrow \quad \Pi = 3 \cdot 0.9 \cdot 1 \cdot 0.9 + 2 \cdot (0.9)^2 \cdot 0.8 \cdot 0.25 + 1 \cdot (0.9)^3 \cdot 0.8 \cdot (0.75) \cdot (0.5) = 1.0827$$

$$\{Z > \Pi\} = \{K=0\} \cup \{K=1\}$$

$$\Rightarrow \quad Pr(Z > \Pi) = \underbrace{Pr(K=0)}_{=q_x} + \underbrace{Pr(K=1)}_{P_x \cdot q_{x+1}} = 0.4$$

Prob. 2: $\frac{d}{dx}\bar{A}_x - (\delta + \mu_x)\bar{A}_x = -\mu_x$

$$\int_0^h (\ldots)\,dx \overset{\text{prod. rule}}{\Longleftrightarrow} \frac{d}{dx}\left(e^{-\int_0^x (\delta+\mu_s)\,ds}\,\bar{A}_x\right) = -\mu_x\, e^{-\int_0^x (\delta+\mu_s)\,ds}$$

$$\Rightarrow \quad e^{-\int_0^h (\delta+\mu_s)\,ds}\,\bar{A}_h - \bar{A}_0 = -\int_0^h \mu_x\, e^{-\int_0^x (\delta+\mu_s)\,ds}\,dx$$

$$\Rightarrow \quad \bar{A}_h = e^{\int_0^h (\delta+\mu_s)\,ds}\,\bar{A}_0 - \int_0^h \mu_x\, e^{\int_x^h (\delta+\mu_s)\,ds}\,dx$$

Ex. 3

Prob. 3: whole life annuity-due

$\longrightarrow$ PV $\quad Y = \sum_{j=0}^{K} v^j$

$\longrightarrow$ net sing. prem.:

$\ddot{a}_x := E[Y] = \sum_{j\geq 0} v^j \cdot {}_jp_x$

$\Rightarrow \quad \ddot{a}_{x+1} = \sum_{j\geq 0} v^j \cdot {}_jp_{x+1} \qquad (*)$

Recall:

$${}_{j-1}p_{x+1} \overset{\text{def.}}{=} \frac{{}_jp_x}{p_x} \iff {}_jp_x = p_x \cdot {}_{j-1}p_{x+1} \qquad (**)$$

$\Rightarrow \quad \ddot{a}_x = \sum_{j\geq 0} v^j {}_jp_x = 1 + \sum_{j\geq 1} v^j \cdot {}_jp_x$

$\overset{(**)}{=} 1 + \sum_{j\geq 1} v^j \cdot p_x \cdot {}_{j-1}p_{x+1} = 1 + \sum_{l\geq 0} v^{l+1} \cdot p_x \cdot {}_lp_{x+1}$

$= 1 + v \cdot p_x \Big(\underbrace{\sum_{l\geq 0} v^l \cdot {}_lp_{x+1}}_{\overset{(*)}{=} \ddot{a}_{x+1}} \Big) \qquad (***)$

p_{73} ?

Use $(***)$, that is

$\ddot{a}_x = 1 + v \cdot \ddot{a}_{x+1} \cdot p_x$

$\Rightarrow \quad \ddot{a}_{73} = 1 + v \cdot \ddot{a}_{74} \cdot p_{73}$

$v = \frac{1}{1+0.03}, \quad \ddot{a}_{73} = 7.73, \quad \ddot{a}_{74} = 7.43$

$\Rightarrow \quad p_{73} = \frac{6.73}{7.43} \cdot 1.03 \approx 0.93$

Prob. 4: Def.

$PV_{j+1} := \sum_{l=0}^{j} (2+l) v^l, \quad j = 0, 1, 2$

$\longrightarrow$ PV of the 3 year temp. life ann.-due

$Y = PV_{K+1} 1_{\{K<3\}} + PV_3 1_{\{K\geq 3\}}$

Var[Y] ?

(3)

Ex. 3

$$Var[Y] = E[(Y - E[Y])^2]$$
$$= E[Y^2] - (E[Y])^2$$

1. $E[Y] = \sum_{j=0}^{2} PV_{j+1} \cdot \underbrace{Pr(K=j)}_{{}_jp_x \cdot \underbrace{q_{x+j}}_{(1-p_{x+j})}} + PV_3 \cdot \underbrace{Pr(K \geq 3)}_{= {}_3p_x}$

$$= \sum_{j=0}^{2} PV_{j+1} \cdot {}_jp_x (1 - p_{x+j}) + PV_3 \cdot {}_3p_x$$

$$\left.\begin{array}{l} {}_2p_x = p_x \cdot p_{x+1} \\ {}_3p_x = \underbrace{p_x \cdot p_{x+1}}_{= {}_2p_x} \, p_{x+2} \end{array}\right\} \text{Exerc. 2, Prob. 1}$$

$\Rightarrow$ ${}_2p_x = 0.8 \cdot 0.75 = 0.6$

${}_3p_x = 0.6 \cdot 0.5 = 0.3$

$PV_1 = 2$, $PV_2 = 2 + 3 \cdot \underbrace{0.9}_{=v} = 4.7$

$PV_3 = 2 + 3 \cdot 0.9 + 4 \cdot (0.9)^2 = 7.94$

$\Rightarrow$ $E[Y] = 2 \cdot 1 \cdot (1 - 0.8) + 4.7 \cdot 0.8 (1 - 0.75) +$
$7.94 \cdot 0.3 + 7.94 \cdot 0.6 \cdot (1 - 0.5) = 6.104$

2. $E[Y^2] = \sum_{j=0}^{2} (PV_{j+1})^2 \, {}_jp_x (1 - p_{x+j}) + (PV_3)^2 \, {}_3p_x$

$= 2^2 \cdot 1 \cdot (1 - 0.8) + (4.7)^2 \cdot 0.8 (1 - 0.75)$
$+ (7.94)^2 \cdot 0.6 (1 - 0.5) + (7.94)^2 \cdot 0.3$
$= 43.04416$

$\Rightarrow Var[Y] \approx 5.785$

STK 2520
Solution to Exerc. 3, Prob. 5

Problem 5 (i) We know that

$$_tp_x = \exp(\int_0^t \mu_{x+s} ds).$$

So

$$_{s_i - t_i}p_{x+t_i} = \exp(\int_0^{s_i - t_i} \mu_{x+t_i+s} ds) = \exp(\int_{t_i}^{s_i} \mu_{x+s} ds).$$

hence we get

$$L = (\mu_{x+\frac{1}{2}})^{D_x} \exp(-\mu_{x+\frac{1}{2}} E_x).$$

Then maximization of L with respect to $\mu_{x+\frac{1}{2}}$ gives

$$\widehat{\mu}_{x+\frac{1}{2}} = \frac{D_x}{E_x}$$

(ii) Suppose for example that one of the two lives $(x + 0.4)$ dies at $x + 0.6$. This implies that the other live of the two either leaves at $x + 0.5$, $x + 0.8$ or $x + 1$. So the exposure must be ~~either~~

$$\begin{aligned} E_x &= ((x+0.6) - (x+0.4)) + ((x+0.5) - (x+0.4)) + ((x+0.2) - x) + 2 \times ((x+0.8) - x)) \\ &\quad +7 \times ((x+1) - x)) \\ &= 9.1. \end{aligned}$$

All the other possibilities must also give the same value, that is

$$E_x = 9.1.$$

Since

$$\widehat{\mu}_{x+\frac{1}{2}} = \frac{D_x}{E_x}$$

and $D_x = 1$ we see that

$$\widehat{\mu}_{x+\frac{1}{2}} = \frac{1}{9.1} = 0.1098901.$$