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Exercises 4

Problem 1:

$$Z = v^{K+1}$$

$\longrightarrow$ PV of a paym. 1 at $t = K+1$

$$V = G \cdot \ddot{a}_{\overline{K+1|}}$$ ↙ whole life annuity-due

$\longrightarrow$ PV of annu. prem. G from $t = 0, \dots, K$

Consider

$$L = Z - V \quad , \text{if } G = P_x$$

and

$$L^* = Z - V \quad \text{with } G \text{ s.t.}$$

$$E[L^*] = -0.20$$

$$\ddot{a}_{\overline{K+1|}} = \frac{1 - v^{K+1}}{d} \qquad (1)$$

$$\Rightarrow \quad L = \left(1 + \frac{P_x}{d}\right) v^{K+1} - \frac{P_x}{d} \qquad \Rightarrow$$

$$\underbrace{Var[L]}_{= 0.30} = \left(1 + \frac{P_x}{d}\right)^2 Var[v^{K+1}] \qquad (2)$$

Similarly

$$Var[L^*] = \left(1 + \frac{G}{d}\right)^2 Var[v^{K+1}] \qquad (3)$$

On the other hand

$$\underbrace{E[L]}_{=0} = A_x - P_x \ddot{a}_x \overset{(1)}{=} (1 - d\ddot{a}_x) - P_x \ddot{a}_x$$

$$= 1 - \left(1 + \frac{P_x}{d}\right) d\, \ddot{a}_x \qquad (4)$$

Similarly

$$\underbrace{E[L^*]}_{= -0.2} = 1 - \left(1 + \frac{G}{d}\right) d\, \ddot{a}_x \qquad (5)$$

$$(4),(5) \Rightarrow \quad \frac{\left(1 + \frac{G}{d}\right)}{\left(1 + \frac{P_x}{d}\right)} = \frac{1 + 0.2}{1} = 1.2$$

Further (2),(3) give

$$\frac{Var[L^*]}{\underbrace{Var[L]}_{= 0.30}} = (1.2)^2 \Rightarrow Var[L^*] = 1.44 \cdot 0.3 = 0.432$$

② **Problem 2:** Q ann. prem.

$$Z = \left(\frac{Q}{2} + 10.000\right) v^{K+1}\, 1_{\{K<20\}} + 10.000\, v^{K+1}\, 1_{\{K\geq 20\}}$$

$\longrightarrow$ PV of a paym. of $\left(\frac{Q}{2} + 10.000\right)$ at $t = K+1$, if $K < 20$

+ PV of a paym. of 10.000 at $t = K+1$, if $K \geq 20$

$$V = Q\, \ddot{a}_{\overline{K+1|}}\, 1_{\{K<20\}} + Q\, \ddot{a}_{\overline{20|}}\, 1_{\{K\geq 20\}}$$

$\longrightarrow$ PV of ann. prem. Q until $t = \min(19, K)$

$$Z = \frac{Q}{2} v^{K+1}\, 1_{\{K<20\}} + 10.000\, v^{K+1}$$

$$v^n = 1 - d\, \ddot{a}_{\overline{n|}} \quad (*)$$

$$\Rightarrow Z = \frac{Q}{2} 1_{\{K<20\}} + 10.000\, v^{K+1} - \frac{Q \cdot d}{2} \ddot{a}_{\overline{K+1|}}\, 1_{\{K<20\}}$$

$$\Rightarrow L = Z - V = \frac{Q}{2} 1_{\{K<20\}} + 10.000\, v^{K+1} - Q\left(1 + \frac{d}{2}\right) \ddot{a}_{\overline{K+1|}}\, 1_{\{K<20\}} - Q\, \ddot{a}_{\overline{20|}}\, 1_{\{K\geq 20\}}$$

$$\overset{(*)}{=} \frac{Q}{2}\left(1 - v^{20}\, 1_{\{K\geq 20\}}\right) + 10.000\, v^{K+1} - Q\left(1 + \frac{d}{2}\right)\left(\ddot{a}_{\overline{K+1|}}\, 1_{\{K<20\}} + \ddot{a}_{\overline{20|}}\, 1_{\{K\geq 20\}}\right)$$

$$\overset{E[\cdot]}{\Longrightarrow} 0 = E[L] = \frac{Q}{2}\left(1 - v^{20}\, {}_{20}p_x\right) + 10.000\, A_x - Q\left(1 + \frac{d}{2}\right) \ddot{a}_{x:\overline{20|}}$$

Problem 3: 9-year immediate life ann.

$$\longrightarrow Y = a_{\overline{K|}}\, 1_{\{K<9\}} + a_{\overline{9|}}\, 1_{\{K\geq 9\}}$$

$\longrightarrow$ PV of an ann. paym of 1 from $t = 1, \dots, \min(9, K)$

$$\Rightarrow a_{30:\overline{9|}} := E[Y] = \ddot{a}_{30:\overline{10|}} - 1$$

$$A_{30:\overline{10|}} = \underbrace{E\left[v^{K+1}\, 1_{\{K<10\}}\right]}_{= A^1_{30:\overline{10|}} \text{ term insurance}} + \underbrace{E\left[v^n\, 1_{\{K\geq 10\}}\right]}_{= A_{30:\overline{10|}}^{\ \ 1} \text{ pure endowment}} (*)$$

$$= 1 - d\, \ddot{a}_{30:\overline{10|}}$$

since $\ddot{a}_{\overline{n|}} = \frac{1 - v^n}{d}$

Ex. 4

③ $d = \frac{i}{1+i}$

$\Rightarrow A_{30:\overline{10|}} = 0.4$

$\overset{(*)}{\Rightarrow} A^{1}_{30:\overline{10|}} = A_{30:\overline{10|}} - v^{10}\,{}_{10}p_{30} = 0.4 - 0.035 = 0.05$

$\Rightarrow 1000\, P^{1}_{30:\overline{10|}} = 1000 \frac{A^{1}_{30:\overline{10|}}}{\ddot{a}_{30:\overline{10|}}} = 7.58$

Problem 4 : Recall that $\qquad$ Ex. 4

(4) $\cdot {}_jL$ = PV of the future benefit paym. at time $t=j$
$\qquad$ − PV of the future premium paym. at time $t=j$

${}_0L = L$

${}_jV$ net premium reserve def. by

$${}_jV = E[{}_jL \mid T > j]$$

To make the ins. policy interesting for the insured and the insurer reasonable to require that

1. ${}_0V = E[L] = 0$ (equiv. principle)
2. ${}_jV \geqslant 0$ for all j

3-year endowment with paym. C_{K+1} endowment amount

PV $Z = C_{K+1} v^{K+1} 1_{\{K<3\}} + {}_3V \cdot v^3 1_{\{K \geqslant 3\}}$

recursion formula:

$${}_jV + \Pi_j = v\left[C_{j+1} \cdot q_{x+j} + {}_{j+1}V \cdot p_{x+j} \right]$$

for $\Pi_0 = \Pi_1 = \Pi_2 = 1$, $\Pi_3 = -{}_3V$, $\Pi_4 = \Pi_5 = \ldots = 0$

and $C_1 = 2$, $C_2 = 3$, $C_3 = 4$, $C_4 = C_5 = \ldots = 0$ (see Rem. 7.2

$$\underbrace{{}_0V}_{0} + \underbrace{\Pi_0}_{1} = 0.9 \cdot \left[2 \cdot 0.20 + {}_1V \cdot (1 - 0.20) \right]$$

$\Rightarrow$ ${}_1V = 0.8889$

$\Rightarrow$ ${}_2V = 1.7984$

$\Rightarrow$ ${}_3V = 2.2187$

(5)

Ex. 4

Problem 5: ${}_2V_{x:\overline{4|}}$?

PV of 4-year endowment:

$$Z = v^{K+1} 1_{\{K<4\}} + v^4 \cdot 1_{\{K \geq 4\}}$$

$\longrightarrow$ PV of a term ins. + PV of a pure endowment

PV of premiums:

$$V = \Pi \cdot \ddot{a}_{\overline{K+1|}} 1_{\{K<4\}} + \Pi\, \ddot{a}_{\overline{4|}} 1_{\{K \geq 4\}}$$

$\longrightarrow$ PV of annual paym. of the prem. Π from $t = 0, \dots, \min(3, K)$

$\Longrightarrow$ V PV of 4-year temp. life annuity-due

An endowment is a special case of a general type of life insurance with

$$L = c_{K+1} v^{K+1} - \sum_{j=0}^{K} \Pi_j v^j$$

for $c_1, c_2, \dots, c_n = 1$, $c_{n+1} = c_{n+2} = \dots = 0$

and $\Pi_0 = \Pi_1 = \dots = \Pi_{n-1} = P_{x:\overline{n|}}$, $\Pi_n = -1$, $\Pi_{n+1} = \Pi_{n+2} = \dots = 0$

$\longrightarrow$ recursion:

$${}_kV + \Pi_k = c_{k+1} v + ({}_{k+1}V - c_{k+1}) v\, p_{x+k} \qquad (*)$$

See Section 6 in Gerber.

$$P_{x:\overline{4|}} = \frac{A_{x:\overline{4|}}}{\ddot{a}_{x:\overline{4|}} = E[V]}$$

net single prem. of the endowm.

$$\ddot{a}_{x:\overline{4|}} = \sum_{j=0}^{3} \ddot{a}_{\overline{j+1|}}\, {}_jp_x \cdot q_{x+j} + \ddot{a}_{\overline{4|}}\, {}_4p_x \qquad (**)$$

On the other hand

$$\ddot{a}_{x:\overline{4|}} = \frac{1 - A_{x:\overline{4|}}}{d} \qquad (***)$$

$$d = \frac{i}{1+i}$$

$$\ddot{a}_{\overline{2|}} = 1 + v = 1.930 \overset{v = \frac{1}{1+i}}{\Longrightarrow} i \approx 0.0753$$

$\Longrightarrow d \approx 0.070$

Recall ${}_{k-1|}q_x = \Pr(K-1 \leq T \leq K) = {}_{k-1}p_x \cdot q_{x+k-1}$

$$\Longrightarrow \sum_{l=1}^{K} {}_{l-1|}q_x = {}_kq_x \qquad \Longrightarrow$$

Ex. 4

⑥ $q_{x+k-1} = \frac{{}_{k-1|}q_x}{{}_{k-1}p_x}$

with ${}_kp_x = 1 - \sum_{l=1}^{k} {}_{l-1|}q_x$

$\Rightarrow$ $p_x = 0.67$, ${}_2p_x = 0.43$, ${}_3p_x = 0.27$, ${}_4p_x = 0.16$

$q_x = 0.33$, $q_{x+1} \approx 0.358$, $q_{x+2} \approx 0.372$, $q_{x+3} \approx 0.4074$

$\overset{(**)}{\Longrightarrow}$ $\ddot{a}_{x:\overline{4|}} \approx 2.2121$

$\overset{(***)}{\Longrightarrow}$ $A_{x:\overline{4|}} \approx 0.845$ $\Longrightarrow$ $P_{x:\overline{4|}} \approx 0.382$

$\overset{(*)}{\Longrightarrow}$ ${}_1V \approx 0.1205$ $\overset{(*)}{\Longrightarrow}$ ${}_2V = {}_2V_{x:\overline{4|}} \approx 0.284$

(7) Problem 6 :

Recall : The total loss Λ_j of the insurer during year $j+1$ is given by

$$\Lambda_j = (\underbrace{C_{j+1} v}_{\text{PV at } t=j \text{ of the paym. } C_{j+1} \text{ at } j+1} - ({}_jV + \Pi_j)) 1_{\{K=j\}}$$

$$+ (\underbrace{{}_{j+1}V \cdot v}_{\text{PV at } t=j \text{ of the reserve } {}_{j+1}V \text{ at } j+1} - ({}_jV + \Pi_j)) \cdot 1_{\{K \geq j+1\}}$$

Note that

$E[\Lambda_j] = 0$ (Section 6.7 in Gerber)

$$\Rightarrow \quad Var[\Lambda_1] = E[\Lambda_1^2]$$

$$= (\underset{3}{C_2}\, \underset{0.9}{v} - (\underset{0.8889}{{}_1V} + \underset{1}{\Pi_1}))^2 \overbrace{Pr(K=1)}^{= {}_1p_x \cdot q_{x+1} = 0.8 \cdot 0.25}$$

$$+ (\underset{1.7984}{{}_2V} \cdot \overset{0.9}{v} - (\underset{0.8889}{{}_1V} + \underset{1}{\Pi_1}))^2 \underbrace{Pr(K \geq j+1)}_{{}_2p_x = \underbrace{p_x}_{0.8} \cdot \underbrace{p_{x+1}}_{=(1-\underbrace{q_{x+1}}_{})=0.75}}$$

$$\Rightarrow Var[\Lambda_1] \approx 0.1754$$