1

N-L Insurance

Exercises 5

Prob. 1 : ${}_5V$?

Moivre : $g(t) = \frac{1}{105-x} = \frac{1}{30}, \; 0 \le t \le 105 - x = 30$ (+)

$$\Rightarrow \quad p_{x+K} = 1 - \int_0^1 \frac{1}{105-(x+K)}\,ds \overset{x=75}{=} \frac{30-(K+1)}{30-K}$$

$$L = 1000\, v^{K+1} - \sum_{\ell=0}^{K} \underbrace{\Pi_\ell}_{=\Pi_0} \cdot v^\ell \quad \text{total loss}$$

$$E[L] = 0 \quad (\text{equiv. principle})$$

$$\Leftrightarrow \quad 0 = 1000\, A_{75} - E[K+1]\cdot \Pi_0 \qquad (++)$$

$$A_{75} = E[v^{K+1}] \overset{(+)}{=} \int_0^{30} v^{[y]+1} \cdot \frac{1}{30}\,dy = \sum_{\mu=0}^{30-1} \int_\mu^{\mu+1} v^{\overbrace{[y]}^{\mu \text{ on } [\mu,\mu+1)}+1} \frac{1}{30}\,dy$$

NSP of the whole life insurance

$$= \frac{1}{30} \sum_{\mu=0}^{29} v^{\mu+1} = \frac{1}{30} v \cdot \frac{1-v^{30}}{1-v} \approx 0.513$$

$$\underbrace{E[K+1]}_{\overset{\text{def}}{=} e_x + 1} \overset{(+)}{=} \int_0^{30} ([y]+1)\cdot\frac{1}{30}\,dy = \frac{1}{30}\sum_{\mu=0}^{29}(\mu+1) = \frac{1}{30}\left(\frac{29\cdot 30}{2} + 30\right) =$$

$$= 15.5$$

$$\overset{(++)}{\Longrightarrow} \quad \Pi_0 = \frac{1000\, A_{75}}{15.5} \approx 33.097$$

recursion $\Longrightarrow$ ${}_1V \approx 1.4674$, ${}_2V \approx 3.6742$, ${}_3V \approx 6.6967$, ${}_4V \approx 10.6173$,

$\underline{{}_5V \approx 15.525}$

Exercises 5

Prob. 2 : (i) Calculate the maximum likelihood estimator $\hat{\lambda}$ based on the Danish fire insurance data from 02/09/1987 until 02/22/1987, that is maximize the likelihood function

$$L(\lambda) = \prod_{j=1}^{n} f_{W_i}(\lambda; W_j) \quad \text{w.r.t. } \lambda > 0$$

with i.i.d inter-arrival times $W_i \sim Exp(\lambda)$

$$\Rightarrow \quad L(\lambda) = \prod_{j=1}^{n} \lambda \exp(-\lambda W_j) = \lambda^n \exp(\lambda(\underbrace{W_1 + \dots + W_n}_{=T_n}))$$

$\Rightarrow$ L attains max. in $\hat{\lambda} = \frac{n}{T_n}$

data $\rightarrow$ $n = 10$, $T_n = 13$

$$\Rightarrow \quad \hat{\lambda} = \frac{10}{13} \approx 0.7692$$

(ii) distr. of T_n ?

Since $T_n = W_1 + \dots + W_n$ with i.i.d $W_i \sim Exp(\hat{\lambda})$ we get that

$$P(T_n \leq x) = 1 - e^{-\hat{\lambda}x} \sum_{k=0}^{n-1} \frac{(\hat{\lambda}x)^k}{k!}, \quad x \geq 0 \qquad (*)$$

Consider e.g. $n = 2$:

$$P(T_2 \leq x) = P(W_1 + W_2 \leq x) = E[\, 1_{\{0 \leq w_1 + w_2 \leq x\}}(W_1, W_2)\,]$$

$$= E[\, F(W_1, W_2)\,]$$

where $F(w_1, w_2) = 1_{\{0 \leq w_1 + w_2 \leq x\}}(w_1, w_2)$ and where (W_1, W_2) has joint density

$$g(w_1, w_2) = \hat{\lambda} e^{-\hat{\lambda} w_1} \cdot \hat{\lambda} e^{-\hat{\lambda} w_2}$$

$$\Rightarrow \quad P(T_2 \leq x) = \int_0^\infty \int_0^\infty F(w_1, w_2)\, g(w_1, w_2)\, dw_1\, dw_2$$

$$= \int_0^x e^{-\hat{\lambda} w_2} \int_0^{x - w_2} \hat{\lambda}^2 e^{-\hat{\lambda} w_1}\, dw_1\, dw_2$$

$$= 1 - e^{-\hat{\lambda}x} \left(\sum_{k=0}^{1} \frac{(\hat{\lambda}x)^k}{k!} \right)$$

$$\xRightarrow{(*)} \quad f_{T_n}(x) = e^{-\hat{\lambda}x} \hat{\lambda}^n \frac{x^{n-1}}{\underbrace{\Gamma(n)}_{=(n-1)!}}, \quad x > 0 \quad \Rightarrow$$

T_n Gamma distr. with shape parameter $\alpha = n$ and scale param. $\lambda = \hat{\lambda}$

3

$$\Rightarrow \quad E[T_n] = \frac{\alpha}{\lambda} = \frac{n}{\lambda} = \frac{13 \cdot n}{10} \quad \text{and}$$

$$Var[T_n] = \frac{\alpha}{\lambda^2} = \frac{n}{\lambda^2} = \left(\frac{13}{10}\right)^2 \cdot n$$

$$\overset{n=1}{\Rightarrow} \quad E[W_1] = 1.3 \quad \text{expected inter-arrival time}$$

Prob. 3 : We have to show that

$$P(B(t) \le x_1, F(t) \le x_2) = P(B(t) \le x_1) \cdot P(F(t) \le x_2), \quad x_1, x_2 \geq 0$$

Let $x_1 < t$ then

$$\{B(t) \le x_1\} \underset{T_{N(t)} \le t}{\overset{\text{def. of } B(t),}{=}} \{t - x_1 \le T_{N(t)} \le t\}$$

$$= \{N(t-x_1, t] \geq 1\} = \text{at least one jump of } S \text{ on } (t-x_1, t] \quad (+)$$

$$\overset{(+)}{\Rightarrow} \quad P(B(t) \le x_1) = 1 - P(N(t-x_1, t] = 0) = 1 - e^{-\lambda x_1}$$

If $x_1 \geq t$ then

$$P(B(t) \le x_1) = P(\underbrace{t - x_1}_{\le 0} \le T_{N(t)} \le t) = 1$$

On the other hand

$$\{F(t) \le x_2\} \underset{T_{N(t)+1} > t}{\overset{\text{def. of } F(t),}{=}} \{t < T_{N(t)+1} \le t + x_2\}$$

$$= \{N(t, t+x_2] \geq 1\}, \quad x_2 > 0 \qquad (++)$$

$$\overset{(++)}{\Rightarrow} \quad P(F(t) \le x_2) = 1 - e^{-\lambda x_2}, \quad x_2 \geq 0$$

W.l.o.g. let $x_1 < t$, $x_2 > 0$

Then

$$P(B(t) \le x_1, F(t) \le x_2) \overset{(+),(++)}{=} P(N(t-x_1, t] \geq 1, N(t, t+x_2] \geq 1)$$

$$\overset{\substack{N \text{ has} \\ \text{indep.} \\ \text{increm.}}}{=} P(\underbrace{N(t-x_1, t] \geq 1}_{\overset{(+)}{=} P(B(t) \le x_1)}) \cdot P(\underbrace{N(t, t+x_2] \geq 1}_{\overset{(++)}{=} P(F(t) \le x_2)})$$

$\longrightarrow$ Prob. 2 is called <u>inspection paradox</u>, since intuition says that

$$P(B(t) \le x_1) < 1 - e^{-\lambda x_1}, \quad x_1 < t$$

and

$$P(F(t) \le x_2) < 1 - e^{-\lambda x_2}, \quad x_2 > 0$$