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N-L Insurance

Exercises 6

Prob. 1 : Recall from Ex. 5, Prob. 1 that

$\hat{\lambda} = \frac{n}{T_n}$ ← sample size — is the maximum likelihood estimator for the intensity

SLLN →

$$\frac{1}{\hat{\lambda}} = \frac{1}{n} T_n = \frac{1}{n} \sum_{i=1}^{n} W_i \xrightarrow[n\to\infty]{} E[W_1]$$

$\Longrightarrow$ $\hat{\lambda} \approx 1/$ sample mean of W_i in a certain period

intensities $\hat{\lambda}$ vary over time

→ reasonable : $N(t)$ modelled by an inhom. Poisson proc. with

$$\mu(t) = (t-i+1)\hat{\lambda}_{1980+i-1} + \sum_{j=1}^{i-1} \hat{\lambda}_{1980+j-1}, \text{ if } i-1 \le t < i, \; i = 1, \dots, 11 \quad (+$$

→ $E[\tilde{S}(t)]$?

where

$$\tilde{S}(t) = \sum_{i=1}^{N(t)} e^{-r(t-T_i)} X_i$$

← discount factor

$r = 0.03$, $t = 10$ years, $E[X_1] = 2.2$

We know that

$$N(t) \stackrel{d}{=} \hat{N}(\mu(t))$$

and

$$T_i \stackrel{d}{=} \mu^{-1}(\hat{T}_i)$$

for the arrival times $\hat{T}_i$ of a hom. Poisson proc. $\hat{N}$ with $\lambda = 1$

$$\Longrightarrow \tilde{S}(t) = \sum_{i=1}^{\hat{N}(\mu(t))} e^{-r\cdot\mu^{-1}(\hat{T}_i)} \cdot X_i$$

$$= \sum_{i=1}^{\hat{N}(\mu(t))} g(\hat{T}_i, X_i)$$

Prop. 2.1.16 in Mikosch $\Longrightarrow$

$$\tilde{S}(t) \stackrel{d}{=} \sum_{i=1}^{\hat{N}(\mu(t))} g(\mu(t)\cdot U_i, X_i)$$

← distribution

$$= \sum_{i=1}^{\hat{N}(\mu(t))} e^{-r\cdot\mu^{-1}(\mu(t)\cdot U_i)} \cdot X_i$$

for iid $U_i \sim U(0,1)$ indep. of $X_i, \hat{T}_i$

$$\Longrightarrow E[\tilde{S}(t)] = E\Big[\sum_{i=1}^{\hat{N}(\mu(t))} e^{-r\cdot\mu^{-1}(\mu(t)\cdot U_i)} \cdot X_i\Big]$$

(2)

$$= E\Big[\sum_{n\geq 0}\Big(\sum_{i=1}^{n} e^{-r\cdot\mu^{-1}(\mu(t)U_i)}\, X_i\Big)\cdot 1_{\{\hat N(\mu(t))=n\}}\Big]$$

$$= \sum_{n\geq 0} E\Big[\Big(\sum_{i=1}^{n} e^{-r\cdot\mu^{-1}(\mu(t)U_i)}\, X_i\Big)\cdot 1_{\{\hat N(\mu(t))=n\}}\Big]$$

indep. r.v.'s

$$= \sum_{n\geq 0} E\Big[\Big(\sum_{i=1}^{n} e^{-r\cdot\mu^{-1}(\mu(t)U_i)}\, X_i\Big]\cdot \underbrace{E\big[1_{\{\hat N(\mu(t))=n\}}\big]}_{P(\hat N(\mu(t))=n)}$$

indep. r.v.'s

$$= \sum_{n\geq 0} P(\hat N(\mu(t))=n) \sum_{i=1}^{n} E\big[e^{-r\cdot\mu^{-1}(\mu(t)U_i)}\big]\, E[X_i]$$

$$\overset{U_i, X_i \text{ i.i.d.}}{=} \underbrace{\Big(\sum_{n\geq 0} n\cdot P(\hat N(\mu(t))=n)\Big)}_{=E[\hat N(\mu(t))]=\mu(t)} \underbrace{E\big[e^{-r\cdot\mu^{-1}(\mu(t)U_1)}\big]}_{=:J}\cdot E[X_1] \qquad (*)$$

$t=10 \overset{(+)}{\Rightarrow} \mu(10) = 5.34\ (\times 365\text{ days})$

$$J = \int_0^1 e^{-0.03\cdot\mu^{-1}(5.34\cdot x)}\,dx \overset{x=\frac{\mu(u)}{5.34}}{=} \frac{1}{5.34}\int \overbrace{\sum_{i=1}^{10} \lambda_{1980+i-1} 1_{[i-1,i)}(u)}^{=\mu'(u)} \cdot e^{-0.03\cdot u}\,du = \frac{1}{5.34}\sum_{i=1}^{10} \lambda_{1980+i-1}\Big(-\frac{1}{0.03} e^{-0.03\cdot u}\Big|_{u=i-1}^{i}\Big)$$

$$\approx 0.85459$$

$$\overset{(*)}{\Rightarrow} E[S(t)] \approx 5.34\cdot 365\cdot 0.85459\cdot 2.2 \approx 3664.5$$

Prob. 2 : $E[\hat N(t)]$?

where $\hat N(t) = \sum_{i=1}^{N(t)} 1_{[0,t]}(T_i+V_i)$

$\longrightarrow$ # of claims reported up to time t

V_i delays i.i.d $U(0,1)$ indep. of (T_i)

$\lambda = 1$

Regard V_i as claim X_i

$$\overset{\text{Prop. 2.1.16 in Mikosch}}{\Rightarrow} \quad \hat N(t) \overset{d}{=} \sum_{i=1}^{N(t)} 1_{[0,t]}(t\cdot U_i+V_i)$$

with i.i.d $U_i \sim U(0,1)$ indep. of (T_i), (V_i)

(3)

$$\Rightarrow \quad E[\hat{N}(t)] = E\Big[\sum_{i=1}^{N(t)} 1_{[0,t]}(t\cdot U_i + V_i)\Big]$$

$$= E\Big[\sum_{n\geq 0}\Big(\sum_{i=1}^{N(t)} 1_{[0,t]}(t\cdot U_i + V_i)\Big)\cdot 1_{\{N(t)=n\}}\Big]$$

indep.

$$= \sum_{n\geq 0} \underbrace{E\Big[\sum_{i=1}^{n} 1_{[0,t]}(t\cdot U_i + V_i)\Big]}_{= n\cdot E[1_{[0,t]}(t\cdot U_1 + V_1)]}\cdot P(N(t)=n)$$

$$= \underbrace{E[N(t)]}_{=t}\cdot E[\underbrace{1_{[0,t]}(t\cdot U_1 + V_1)}_{=:\,J}]$$

$$J = \int_0^1\int_0^1 1_{[0,t]}(t\cdot x_1 + x_2)\, dx_1\, dx_2$$

$$= \int_0^{1\wedge t} \frac{1}{t}(t - x_2)\, dx_2 = (1\wedge t) - \frac{1}{t}\frac{(1\wedge t)^2}{2}$$

$$\Rightarrow \quad E[\hat{N}(t)] = t\cdot\Big((1\wedge t) - \frac{1}{t}\frac{(1\wedge t)^2}{2}\Big)$$

$$\Rightarrow \quad E[\hat{N}(t)] \approx t = E[N(t)]$$

for large t

Exerc. 6

Prob. 3 : $P(T_1 \le x_1, T_2 \le x_2) = P(\overbrace{N(x_1)}^{//N(x_2)-N(x_1)+N(x_1)} \geq 1, N(x_2) \geq 2)$

$= E[\, P(N(x_1) \geq 1, N(x_2)-N(x_1)+N(x_1) \geq 2 \mid N(x_1))\,]$

$= E[\sum_{j\geq 0} P(N(x_1)\geq 1, N(x_2)-N(x_1)+N(x_1)\geq 2 \mid N(x_1)=j)\cdot$
$\cdot \mathbb{1}_{\{N(x_1)=j\}}]$

$= E[\sum_{j\geq 0} P(j\geq 1, N(x_2)-N(x_1)+j\geq 2 \mid N(x_1)=j)\cdot$
$\cdot \mathbb{1}_{\{N(x_1)=j\}}]$

$= \sum_{j\geq 0} P(j\geq 1, N(x_2)-N(x_1)+j \geq 2 \mid N(x_1)=j)\, \underbrace{E[\mathbb{1}_{\{N(x_1)=j\}}]}_{=P(N(x_1)=j)}$

$= P(N(x_2)-N(x_1)+1\geq 2)\,P(N(x_1)=1)$
$+ \sum_{j\geq 2} \underbrace{P(j\geq 1, \overbrace{N(x_2)-N(x_1)}^{\geq 0}+j\geq 2)}_{=1 \text{ for } j\geq 2} P(N(x_1)=j)$

$= \underbrace{P(N(x_2)-N(x_1)\geq 1)}_{=1-P(N(x_2)-N(x_1)=0)} P(N(x_1)=1) + \underbrace{\sum_{j\geq 2} P(N(x_1)=j)}_{=P(N(x_1)\geq 2) = 1-P(N(x_1)\le 1) = 1-P(N(x_1)=0)-P(N(x_1)=1)}$

$\overset{\text{Poisson distr.}}{=} (1-e^{-\lambda(x_2-x_1)})e^{-\lambda x_1}\lambda x_1 + 1 - e^{-\lambda x_1} - e^{-\lambda x_1}\lambda x_1$

$= 1 - e^{-\lambda x_2}\lambda x_1 - e^{-\lambda x_1}$

On the other hand,

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(w_1,w_2)\,g(w_1,w_2)\,dw_1\,dw_2 = \int_0^{x_1}\int_0^{x_2-w_2} \lambda e^{-\lambda w_1}\lambda e^{-\lambda w_2}\,dw_1\,dw_2$

$= 1 - e^{-\lambda x_2}\lambda x_1 - e^{-\lambda x_1}.$