STK 3505
Solutions to Exercises 8

Problem 3 We have real-life fire insurance data collected at Copenhagen Reinsurance in the period from 09/01/1990 until 10/01/1990, that is fire losses

$$X_1 = 14.851485, ..., X_8 = 5.726072$$

and inter-arrival times

$$W_1 = 7, ..., W_8 = 1.$$

(i) $p_{Net}(t)$ and $p_{EV}(t)$? for large t in the renewal model:

$$p_{Net}(t) := E[S(t)] \overset{\text{Prop. 3.1.3, Mikosch}}{\approx} t \cdot \frac{E[X_1]}{E[W_1]}$$

and

$$p_{EV}(t) := (1+\rho)E[S(t)] \overset{\text{Prop. 3.1.3, Mikosch}}{\approx} t \cdot (1+\rho) \cdot \frac{E[X_1]}{E[W_1]}$$

for large t, where $\rho = 0.04$.

We use the sample mean $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ to estimate $E[X_1]$ and $E[W_1]$ and get

$$E[X_1] \approx 4.187 \text{ and } E[W_1] \approx 3.75.$$

So

$$p_{Net}(t) \approx 1.117 \cdot t \text{ and } p_{EV}(t) \approx 1.161 \cdot t.$$

(ii) Recall that

$$p_{Var}(t) := E[S(t)] + \alpha Var[S(t)].$$

Find $\alpha > 0$ such that

$$\frac{p_{Var}(t)}{p_{EV}(t)} \underset{t \longrightarrow \infty}{\longrightarrow} 1 .$$

We have that

$$\begin{aligned} \frac{p_{Var}(t)}{p_{EV}(t)} &= \frac{E[S(t)] + \alpha Var[S(t)]}{(1+\rho)E[S(t)]} = \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)}\frac{Var[S(t)]/t}{E[S(t)]/t} \\ &\overset{\text{Prop. 3.1.3, Mikosch}}{\longrightarrow} \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)}\frac{\lambda[Var[X_1] + Var[W_1]\lambda^2(E[X_1])^2]}{\lambda E[X_1]} \\ &= \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)}\frac{[Var[X_1] + Var[W_1]\lambda^2(E[X_1])^2]}{E[X_1]} \overset{!}{=} 1 \end{aligned}$$

for $t \longrightarrow \infty$, where $\lambda = 1/E[W_1]$. Thus

$$\alpha = \frac{\rho E[X_1]}{Var[X_1] + Var[W_1]\lambda^2(E[X_1])^2}$$

Compute the sample variance $\frac{1}{n-1}\sum_{i=1}^{n} Y_i^2 - n\overline{Y}$ of X_1 and W_1 and we get

$$Var[X_1] \approx 20.764 \text{ and } Var[W_1] \approx 5.357.$$

Then

$$\alpha \approx 0.0061.$$

Problem 4 Since the mixing variable θ is independent of $\widetilde{N}(t)$ we can treat it somewhat like a constant and get

$$E[S(t)] = E[N(t)]E[X_1] = E[\theta] \cdot t \cdot E[X_1], \tag{1}$$

where $N(t) = \widetilde{N}(\theta \cdot t)$. We see that

$$Var[S(t)] = E[(S(t))^2] - (E[S(t)])^2. \tag{2}$$

We get that

$$\begin{aligned}
E[(S(t))^2] &= E[(\sum_{i=1}^{N(t)} X_i)^2] = E[\sum_{n\geq 0}(\sum_{i=1}^{n} X_i)^2 1_{\{N(t)=n\}}] \\
&= \sum_{n\geq 0} E[(\sum_{i=1}^{n} X_i)^2 1_{\{N(t)=n\}}] \overset{N, X_i \text{ indep.}}{=} \sum_{n\geq 0} E[(\sum_{i=1}^{n} X_i)^2] \cdot P(N(t) = n) \\
&= \sum_{n\geq 0}(Var[\sum_{i=1}^{n} X_i] + (E[\sum_{i=1}^{n} X_i])^2) \cdot P(N(t) = n) \\
& \overset{X_i \text{ i.i.d}}{=} \sum_{n\geq 0}(n \cdot Var[X_1] + n^2 \cdot (E[X_1])^2) \\
\cdot P(N(t) &= n) \\
&= E[N(t) \cdot Var[X_1] + (N(t))^2 \cdot (E[X_1])^2] \\
&= E[\theta] \cdot t \cdot Var[X_1] + (Var[N(t)] + (E[N(t)])^2)(E[X_1])^2 \\
& \overset{\text{Prob.2}}{=} E[\theta] \cdot t \cdot Var[X_1] \\
& \quad +(Var[\theta] \cdot t^2 + E[\theta] \cdot t + (E[\theta])^2 \cdot t^2)(E[X_1])^2.
\end{aligned}$$

So it follows from (1) and (2) that

$$\begin{aligned}
& Var[S(t)] \\
= \; & E[\theta] \cdot t \cdot Var[X_1] + \\
& (Var[\theta] \cdot t^2 + E[\theta] \cdot t + (E[\theta])^2 \cdot t^2)(E[X_1])^2 - (E[\theta])^2 \cdot t^2 (E[X_1])^2 \\
= \; & E[\theta] \cdot t \cdot (Var[X_1] + (E[X_1])^2) + Var[\theta] \cdot t^2 \cdot (E[X_1])^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{p_{Var}(t)}{p_{EV}(t)} &= \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)} \frac{Var[S(t)]}{E[S(t)]} \\
&= \frac{1}{(1+\rho)} + \frac{\alpha}{(1+\rho)} \frac{E[\theta] \cdot t \cdot (Var[X_1] + (E[X_1])^2) + Var[\theta] \cdot t^2 \cdot (E[X_1])^2}{E[\theta] \cdot t \cdot E[X_1]} \\
& \underset{t \longrightarrow \infty}{\longrightarrow} \infty.
\end{aligned}$$

The latter is in contrast to the renewal model for which the limit exists.