STK 3505
Solutions

Problem 1

(i) Since the claim numbers $N(t), t \geq 0$ in the Cramer-Lundberg model are modeled by a homogeneous Poisson process with intensity $\lambda > 0$, we know that the inter-arrival times $W_i, i \geq 1$ are i.i.d. with common distribution $W_i \sim Exp(\lambda)$. So by Exercises 5, Problem 2 we know that the MLE $\widehat{\lambda}$ is given by

$$\widehat{\lambda} = \frac{n}{T_n}.$$

The observed inter-arrival times are $W_1 = 1$ (i.e. 09/30/1988 is excluded), $W_2 = 3, W_3 = 4..., W_{12} = 2$. So $n = 12$ (sample size) and $T_n = W_1 + ... + W_{11} = 38$ (nth arrival time of a fire loss for $n = 12$) we get

$$\widehat{\lambda} = \frac{n}{T_n} = \frac{12}{38} \approx 0.316.$$

(ii) Since $W_i \sim Exp(\widehat{\lambda})$, $i \geq 1$, we have

$$P(W_1 > 5) = 1 - P(W_1 \leq 5) = 1 - (1 - \exp(-\widehat{\lambda}5)) = \exp(-\widehat{\lambda}5) \approx 0.206.$$

(iii) Since $N(t)$ is independent of the i.i.d. fire losses $X_i, i \geq 1$ we know (see Exercises 6) that

$$\begin{aligned} E[S(t)] &= E[\sum_{i=1}^{N(t)} X_i] = E[N(t)] \cdot E[X_1] \text{ ("Wald's identity")} \\ &= t\widehat{\lambda} \cdot E[X_1] \end{aligned}$$

Since

$$E[X_1] \approx \frac{1}{12} \sum_{i=1}^{12} X_i = 5.70186,$$

we get for $t = 365$ days

$$p_{EV}(t) = (1 + 0.20)E[S(t)] = 1.20 \cdot 365 \cdot 0.316 \cdot 5.70186 = 788.657 \text{ (mio DKK)}.$$

Problem 2 (i) The survival probability is given by

$${}_tp_x = \left(1 - \frac{t}{120 - x}\right)^{1/6}$$

for $0 \leq t < 120 - x$. We know that

$$\mathring{e}_x = \int_0^\infty ({}_tp_x)dt = \int_0^{120-x} ({}_tp_x)dt = \int_0^{120-x} \left(1 - \frac{t}{120 - x}\right)^{1/6} dt.$$

Using substitution for $u = 1 - t/(120 - x)$, we get

$$\mathring{e}_x = (120 - x) \int_0^1 u^{1/6} du = \frac{6}{7}(120 - x).$$

So $\mathring{e}_{30} = 77.143$.

(ii) Use the recursion formula

$$e_x = p_x(1 + e_{x+1}),$$

where $x = 30$ and $e_x = 49.5$. Then

$$e_{x+10} \approx 40.4.$$

Problem 3 The present value of the benefits and of the premiums are given by

$$Z = 15000v^5 \mathbf{1}_{\{K \geq 5\}} + (\sum_{j=0}^{K} (1+i)^{K+1-j} \Pi_j) v^{K+1} \mathbf{1}_{\{K<5\}}$$

and'

$$V = \sum_{j=0}^{\min(4,K)} \Pi_j v^j,$$

respectively. So the expected total loss of the insurer is given by

$$\begin{aligned} E[L] &= 15000v^5 \cdot {}_5 p_x + \Pi \sum_{l=0}^{4} (\sum_{j=0}^{l} (\frac{1.03}{1.03})^j) {}_l p_x \cdot q_{x+l} \\ &\quad - \Pi (\sum_{l=0}^{4} (\sum_{j=0}^{l} (\frac{1.03}{1.03})^j) {}_l p_x \cdot q_{x+l} + \sum_{j=0}^{4} (\frac{1.03}{1.03})^j) \cdot {}_5 p_x). \end{aligned}$$

Using the equivalence principle, i.e. $E[L] = 0$ yields

$$\Pi = \frac{15000v^5}{\sum_{j=0}^{4} (\frac{1.03}{1.03})^j)} = \frac{15000}{5} v^5,$$

which shows that the net premiums are independent of the distribution of the future life time. So we get

$$\Pi_0 = \Pi = 2587.83, \Pi_1 = 2665.46, \Pi_2 = 2745.42, \Pi_3 = 2827.79, \Pi_4 = 2912.62.$$

Problem 4 (i)

$$\mathring{e}_x = E[T] = \int_0^\infty t\lambda e^{-\lambda t} dt = \frac{1}{\lambda} = 53.19 \text{ (years)}.$$

(ii) Since

$$ {}_tq_x = 1 - e^{-\lambda t}, $$

the force of mortality μ_{x+t} for (25) is given by

$$ \mu_{x+t} \stackrel{def}{=} -\frac{d}{dt}\log\left({}_tp_x\right) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda = 0.0188. $$

(iii) We know that

$$ \Pr(S \le t\,|K = k) = \frac{{}_tq_{x+k}}{q_{x+k}}, 0 \le t \le 1, k \ge 0 $$

See e.g. (2.4.6) in Gerber. Further, we see that

$$ {}_tq_{x+k} \stackrel{def}{=} \frac{G(k+t) - G(k)}{1 - G(k)} = \frac{e^{-\lambda k}(1 - e^{-\lambda t})}{e^{-\lambda k}} = 1 - e^{-\lambda t}. $$

So

$$ \Pr(S \le t\,|K = k) = \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda}}. $$

HenceThen differentiation w.r.t. to t on both sides shows that

$$ \frac{d}{dt}\Pr(S \le t) = \frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda}} $$

is the probability density of S. So using integration by parts we get

$$ E[S] = \int_0^1 t\frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda}}dt = \frac{1}{\lambda} - \frac{e^{-\lambda}}{1 - e^{-\lambda}} = 0.498433. $$

Problem 5 We know from Problem 2, Exercises 4 that the net annual premium Q is given by

$$ \frac{15000A_x}{(1 + \frac{d}{2})\ddot{a}_{x:\overline{10|}} - (1 - v^{10} \cdot {}_{10}\, p_x)/2}, $$

where $d = i/(1+i)$ is the discount rate. Using the fact that

$$ A_x = 1 - d\ddot{a}_x $$

and

$$ \ddot{a}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^k \cdot {}_k\, p_x, $$

(see (4.28) and (4.2.12) in Gerber), we get that $Q = 213.707$.

Then we can compute the premium reserves by

$$ {}_kV + \Pi_k = v(c_{k+1}q_{x+k} +_{k+1} V p_{x+k}) $$

for ${}_0V = 0$, $\Pi_k = Q,\ k = 0, ..., 9, \Pi_k = 0,\ k \geq 10, c(k+1) = \begin{cases} \frac{1}{2}Q + 15000, & \text{if } k < 10 \\ 15000 & \text{else} \end{cases}$ and obtain

$$ {}_1V = 206.214, {}_2V = 422.268, {}_3V = 648.545, {}_4V = 885.597, {}_5V = 1133.87. $$

Problem 6 The present value of the benefit payments are given by

$$ Z = \begin{cases} 200000v^{K+1}, & \text{if } K < 25 \\ 100000v^{25}, & \text{if } K \geq 25 \end{cases} $$

and annual premium payments Q are made as long as the policyholder is alive during the contract period.

Using the equivalence principle, we therefore find

$$ Q = \frac{E[Z]}{\ddot{a}_{x:\overline{25|}}}. $$

We get that

$$ E[Z] = 200000 \sum_{k=0}^{24} v^{k+1} \cdot {}_k p_x q_{x+k} + 100000v^{25} \cdot {}_{25} p_x. $$

See (3.2.7), (3.2.10) and (3.2.14) in Gerber or the manuscript. On the other hand, we know that

$$ {}_k p_x = \exp(-\int_0^k \mu_{x+s} ds) = \exp(-\int_x^{x+k} \mu_u du). $$

Hence, by applying numerical integration, we obtain $Q = 3394.11$.

Then we can compute the premium reserves by

$$ {}_kV + \Pi_k = v(c_{k+1} q_{x+k} + {}_{k+1}V p_{x+k}) $$

for ${}_0V = 0$, $\Pi_k = Q,\ k = 0, ..., 24, \Pi_{25} = -100000, \Pi_k = 0,\ k > 25, c(k+1) = \begin{cases} 200000, & \text{if } k < 25 \\ 0 & \text{else} \end{cases}$ and get that

$$ {}_1V = 3080.9, {}_2V = 6233.46, {}_3V = 9458.56, {}_4V = 12757, ..., {}_{24}V = 95026, {}_{25}V = 100000. $$

Problem 7 See the self-explaining hint.